

LONG WAVELENGTH LIMIT FOR THE QUANTUM EULER-POISSON EQUATION

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ABSTRACT. In this paper, we consider the long wavelength limit for the quantum Euler-Poisson equation. Under the Gardner-Morikawa transform, we derive the quantum Korteweg-de Vries (KdV) equation by a singular perturbation method. We show that the KdV dynamics can be seen at time interval of order $O(\epsilon^{-3/2})$. When the nondimensional quantum parameter $H = 2$, it reduces to the inviscid Burgers equation.

1. INTRODUCTION

In this paper, we consider a one-dimensional two species quantum plasma system made by one electronic and one ionic fluid, in the electrostatic approximation [11]. For simplicity, we only consider the continuity and momentum equations and ignore the energy transport equation, which are sufficient to describe the classical ion-acoustic waves [26]. The system is governed by the following equations

$$\begin{cases} \partial_t n_e + \partial_x(n_e u_e) = 0, & (1.1a) \\ \partial_t n_i + \partial_x(n_i u_i) = 0, & (1.1b) \end{cases}$$

$$\begin{cases} \partial_t u_e + u_e \partial_x u_e = \frac{e}{m_e} \partial_x \phi - \frac{1}{m_e n_e} \partial_x P + \frac{\hbar^2}{2m_e^2} \partial_x \left(\frac{\partial_x^2 \sqrt{n_e}}{\sqrt{n_e}} \right), & (1.1c) \end{cases}$$

$$\begin{cases} \partial_t u_i + u_i \partial_x u_i = -\frac{e}{m_i} \partial_x \phi, & (1.1d) \end{cases}$$

$$\begin{cases} \partial_x^2 \phi = \frac{e}{\epsilon_0} (n_e - n_i), & (1.1e) \end{cases}$$

where $n_{e,i}$ are the electronic and ionic number densities, $u_{e,i}$ the electronic and ionic velocities, ϕ the scalar potential, $m_{e,i}$ the electron and ion masses, $-e$ the electron charge, $\hbar = \frac{h}{2\pi}$, where h is Planck's constant and ϵ_0 the vacuum permittivity. The electron fluid pressure $P = P(n_e)$, modeled by the equation of state for a one dimensional zero-temperature Fermi gas, is given by

$$P = \frac{m_e v_{F_e}^2}{3n_0^2} n_e^3, \quad (1.2)$$

where n_0 is the equilibrium density for both electrons and ions, and v_{F_e} is the electrons Fermi velocity, related to the Fermi temperature T_{F_e} by $m_e v_{F_e}^2 = \kappa_B T_{F_e}$, where κ_B is the Boltzmann constant. Throughout this paper, we assume such a cubic law for the electron fluid pressure, which is the most important significant physical case, as pointed out by Jackson [9, 13].

Equations (1.1a) and (1.1b) represent conservation of charge and mass. Equations (1.1c) and (1.1d) account for momentum balance. The third order term in (1.1c), proportional

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to \hbar^2 , takes into account the influence of quantum diffraction effects. However, the motion of ion can be taken as classical in view of the high ion mass in comparison to the electron mass. Accordingly, (1.1d) contains no quantum terms. Finally, (1.1e) is Poisson's equation, describing the self-consistent electrostatic potential.

Take the following rescaling,

$$\begin{aligned}\bar{x} &= \frac{\omega_{p_e} x}{v_{F_e}}, \quad \bar{t} = \omega_{p_i} t, \quad \bar{n}_e = \frac{n_e}{n_0}, \quad \bar{n}_i = \frac{n_i}{n_0}, \\ \bar{u}_e &= \frac{u_e}{c_s}, \quad \bar{u}_i = \frac{u_i}{c_s}, \quad \bar{\phi} = \frac{e\phi}{\kappa_B T_{F_e}},\end{aligned}\tag{1.3}$$

where ω_{p_e} and ω_{p_i} are the corresponding electron and ion plasma frequencies and c_s is the quantum ion-acoustic velocity, given by

$$\omega_{p_e} = \left(\frac{n_0 e^2}{m_e \epsilon_0} \right)^{1/2}, \quad \omega_{p_i} = \left(\frac{n_0 e^2}{m_i \epsilon_0} \right)^{1/2}, \quad c_s = \left(\frac{\kappa_B T_{F_e}}{m_i} \right)^{1/2}.\tag{1.4}$$

In addition, consider nondimensional parameter $H = \hbar \omega_{p_e} / \kappa_B T_{F_e}$. Physically, H is the ratio between the electron plasmon energy and the electron Fermi energy. Using the new variables and dropping bars for simplifying notation, we obtain from (1.1c)

$$\frac{m_e}{m_i} (\partial_t u_e + u_e \partial_x u_e) = \partial_x \phi - n_e \partial_x n_e + \frac{H^2}{2} \partial_x \left(\frac{\partial_x^2 \sqrt{n_e}}{\sqrt{n_e}} \right).\tag{1.5}$$

Since $m_e/m_i \ll 1$, we let the left-hand side of (1.5) to be zero and then integrate about x with the boundary conditions $n_e = 1$, $\phi = 0$ at infinity, to obtain

$$\phi = -\frac{1}{2} + \frac{1}{2} n_e^2 - \frac{H^2}{2\sqrt{n_e}} \partial_x^2 \sqrt{n_e}.\tag{1.6}$$

This last equation is the electrostatic potential in terms of the electron density and its derivatives. Even when the quantum diffraction effects are negligible ($H = 0$), the electron equilibrium is given by a Fermi-Dirac distribution and not by a Maxwell-Boltzmann one.

Applying the rescaling (1.3) to (1.1b), (1.1d) and (1.1e), we have by dropping the bars

$$\begin{cases} \partial_t n_i + \partial_x (n_i u_i) = 0, & (1.7a) \end{cases}$$

$$\begin{cases} \partial_t u_i + u_i \partial_x u_i = -\partial_x \phi, & (1.7b) \end{cases}$$

$$\begin{cases} \partial_x^2 \phi = n_e - n_i, & (1.7c) \end{cases}$$

Equations (1.7a)-(1.7c), together with (1.6), provide a reduced model of four equations with four unknown quantities, n_i , u_i , n_e and ϕ . This reduced model is the basic model to be studied in the following, which will lead to the quantum Korteweg-de Vries (KdV) equation (2.7) under the Gardner-Morikawa transform [4, 31].

Obviously, the reduced system (1.6)-(1.7) admits the homogeneous equilibrium solution $(n_e, n_i, u_i, \phi) = (1, 1, 0, 0)$. Global existence of smooth solutions around the equilibrium is an outstanding difficult problem for the Euler-Poisson problem. Without quantum effects, Guo [6] firstly obtained global irrotational solutions with small velocity for the 3D electron fluid, based on the Klein-Gordon effect. Then, Jang, Li, Zhang and Wu [14, 15, 22] obtained global smooth small solutions for the 2D electron fluid in Euler-Poisson system. Very recently, Guo, Han and Zhang [9] finally completely settled this problem and proved that no shocks form for the 1D Euler-Poisson system for electrons. For Euler-Poisson equation for ions, Guo and Pausader [8] constructed global smooth irrotational solutions with small amplitude for ion dynamics. For the Euler-Poisson system (1.7) with quantum effects, there is no existence result, to the best knowledge of the authors.

To access weakly nonlinear solutions for the quantum ion-acoustic system (1.6)-(1.7), a singular perturbation method can be applied to the weakly nonlinear classical waves, which finally leads to the quantum KdV equation. For details, see Section 2. To this aspect, one may refer to the recent papers [10, 21, 27]. In particular, Guo and Pu established rigorously the KdV limit for the ion Euler-Poisson system in 1D for both the cold and hot plasma case, where the electron density satisfies the classical Maxwell-Boltzmann law. This result was generalized to the higher dimensional case in [27], and the 2D Kadomtsev-Petviashvili-II (KP-II) equation and the 3D Zakharov-Kuznetsov equation are derived for well-prepared initial data under different scalings. Almost at the same time, [21] also established the KdV limit in 1D and the Zakharov-Kuznetsov equation in 3D from the Euler-Poisson system. Han-Kwan [12] also introduced a long wave scaling for the Vlasov-Poisson equation and derived the KdV equation in 1D and the Zakharov-Kuznetsov equation in 3D using the modulated energy method. For other studies for the Euler-Poisson system or related models, the interested readers may refer to [2, 3, 5, 23, 24, 28], to list only a few. For derivation of the KdV equation from the water waves without surface tension, see [29] and the references therein.

In the present paper, we will continue to study the long wavelength limit for the reduced system (1.7) for ions with quantum effects. Under the Gardner-Morikawa transform, the quantum KdV equation is derived when $H > 0$ and $H \neq 2$. But when $H = 2$, the quantum KdV equation (2.7) reduces to the inviscid Burger's equation. The formal derivation of the quantum KdV equation can be found in [11] and is given in the next section. The main interest in this paper is to make such a formal derivation rigorous. To do so, we need to obtain uniform (in ϵ) estimates for the remainders $(n_{eR}^\epsilon, n_{iR}^\epsilon, u_{iR}^\epsilon)$ and then recover the uniform estimates of ϕ_R^ϵ from the relation (1.6). To apply the Gronwall inequality to complete the proof, we define the triple norm

$$\begin{aligned} \|(N_i, N_e, U)\|_\epsilon^2 = & \|(N_i, N_e, U)\|_{H^2}^2 + \epsilon \|(\partial_x^3 N_e, \partial_x^3 U)\|_{L^2}^2 \\ & + \epsilon^2 \|\partial_x^4 N_e\|_{L^2}^2 + \epsilon^3 \|\partial_x^5 N_e\|_{L^2}^2 + \epsilon^4 \|\partial_x^6 N_e\|_{L^2}^2, \end{aligned} \quad (1.8)$$

which depends on the parameter ϵ in the Gardner-Morikawa transform. But we regard H as a fixed constant. After careful computations, we finally close the estimates in this triple norm, which gives uniform (in ϵ) estimates for the remainders (N_i, N_e, U) in H^2 and completes the proof. The main result is stated in Theorem 2.5. Furthermore, this implies that

$$\sup_{[0, \epsilon^{-3/2}\tau]} \left\| \begin{pmatrix} (n_i - 1)/\epsilon \\ (n_e - 1)/\epsilon \\ u_i/\epsilon \end{pmatrix} - KdV \right\|_{H^2} \leq C\epsilon, \quad (1.9)$$

for some $C > 0$ independent of $\epsilon > 0$, for any fixed $\tau > 0$ of order $O(1)$. Here the 'KdV' stands for the first approximation of (n_i, n_e, u_i) under the Gardner-Morikawa transform in (2.1). It shows that the KdV dynamics can be seen at time interval of order $O(\epsilon^{-3/2})$. The result also applies to the case when $H = 2$, where the inviscid Burger's equation is derived.

The results in this paper can be generalized to the following general cases. Firstly, for definiteness, we let the electron pressure satisfies the cubic law in (1.2), but the result in this paper can be generalized to general γ -law, which will lead to a different relation between ϕ and n_e in (1.6). Secondly, the ion momentum equation (1.1d) does not contain ion pressure, which generally depends on ion density with the form $P_i(n_i) = T_i \ln n_i$. This paper corresponds to the cold ion case $T_i = 0$. But the result in this paper can be generalized to general case $T_i > 0$, and indeed, the proof will be slightly simpler since in this case, the system is Friedrich symmetrizable. The result in this paper can be also generalized to the general γ -law of the ion pressure, i.e., when $P_i(n_i) = T_i n_i^\gamma$ for $\gamma \geq 1$. For clarity, we will not

mention these general cases in the rest of the paper and concentrate on the case $P(n_e) \sim n_e^3$ in (1.2) and zero ion temperature case $T_i = 0$.

This paper is organized as follows. In Section 2, we present the formal derivation of the quantum KdV equation (2.7) and state the main result in Theorem 2.5. In Section 3, we present uniform estimates for the remainders in (2.13). The main estimates are stated in Proposition 3.1 and 3.2. Finally, we complete the proof in Section 4.

2. FORMAL EXPANSION AND MAIN RESULTS

2.1. Formal KdV expansion. By the classical Gardner-Morikawa transformation [4, 31]

$$x \rightarrow \epsilon^{\frac{1}{2}}(x - t), \quad t \rightarrow \epsilon^{\frac{3}{2}}t, \quad (2.1)$$

we obtain from (1.7) the parameterized system

$$\begin{cases} \epsilon \partial_t n_i - \partial_x n_i + \partial_x (n_i u_i) = 0, \\ \epsilon \partial_t u_i - \partial_x u_i + u_i \partial_x u_i = -\partial_x \phi, \\ \epsilon \partial_x^2 \phi = n_e - n_i, \end{cases} \quad (2.2)$$

where ϵ is the amplitude of the initial disturbance and is assumed to be small compared with unity and (1.6) is rescaled into the following relation

$$\phi = -\frac{1}{2} + \frac{1}{2}n_e^2 - \frac{\epsilon H^2}{2\sqrt{n_e}} \partial_x^2 \sqrt{n_e}.$$

We consider the following formal expansion around the equilibrium solution $(n_i, n_e, u_i) = (1, 1, 0)$,

$$\begin{cases} n_i = 1 + \epsilon n_i^{(1)} + \epsilon^2 n_i^{(2)} + \epsilon^3 n_i^{(3)} + \epsilon^4 n_i^{(4)} + \dots, \\ n_e = 1 + \epsilon n_e^{(1)} + \epsilon^2 n_e^{(2)} + \epsilon^3 n_e^{(3)} + \epsilon^4 n_e^{(4)} + \dots, \\ u_i = \epsilon u_i^{(1)} + \epsilon^2 u_i^{(2)} + \epsilon^3 u_i^{(3)} + \epsilon^4 u_i^{(4)} + \dots. \end{cases} \quad (2.3)$$

Plugging (2.3) into (1.7), we get a power series of ϵ , whose coefficients depend on $(n_i^{(k)}, n_e^{(k)}, u_i^{(k)})$ for $k = 1, 2, \dots$.

At the order $O(1)$, the coefficients are automatically balanced.

At the order $O(\epsilon)$, we obtain

$$\begin{aligned} (\mathcal{S}_0) \quad & \begin{cases} -\partial_x n_i^{(1)} + \partial_x u_i^{(1)} = 0, & (2.4a) \\ -\partial_x u_i^{(1)} = -\partial_x n_e^{(1)}, & (2.4b) \\ 0 = n_e^{(1)} - n_i^{(1)}. & (2.4c) \end{cases} \end{aligned}$$

This enables us to assume the relation

$$(\mathcal{L}_1) : \quad n_e^{(1)} = n_i^{(1)} = u_i^{(1)}, \quad (2.5)$$

which makes (2.4) valid and shows that the mode is quasi-neutral in a first approximation. Then only $n_i^{(1)}$ needs to be determined.

At the order $O(\epsilon^2)$, we obtain

$$(\mathcal{S}_1) \quad \begin{cases} \partial_t n_i^{(1)} - \partial_x n_i^{(2)} + \partial_x u_i^{(2)} + \partial_x (n_i^{(1)} u_i^{(1)}) = 0, & (2.6a) \end{cases}$$

$$\begin{cases} \partial_t u_i^{(1)} - \partial_x u_i^{(2)} + u_i^{(1)} \partial_x u_i^{(1)} = -\partial_x n_e^{(2)} - n_e^{(1)} \partial_x n_e^{(1)} + \frac{H^2}{4} \partial_x^3 n_e^{(1)}, & (2.6b) \end{cases}$$

$$\begin{cases} \partial_x^2 n_e^{(1)} = n_e^{(2)} - n_i^{(2)}. & (2.6c) \end{cases}$$

Differentiating (2.6c) with respect to x , and then adding the resultant and (2.6a) to (2.6b) together, we deduce that $n_i^{(1)}$ satisfies the quantum Kortweg-de Vries equation

$$\partial_t n_i^{(1)} + 2n_i^{(1)} \partial_x n_i^{(1)} + \frac{1}{2} \left(1 - \frac{H^2}{4}\right) \partial_x^3 n_i^{(1)} = 0, \quad (2.7)$$

where we have used the relation (2.5). We note that the system (2.5), (2.7) are self-contained, which do not depend on $(n_i^{(j)}, n_e^{(j)}, u_i^{(j)})$ for $j \geq 2$. We also note that (2.7) is different from the classical KdV equation due to the presence of the parameter H . When $H \neq 2$, it can be transformed into the classical KdV equation, while when $H = 2$, it reduces to the inviscid Burger's equation, which is drastically different from the KdV equation. For derivation of KdV from water waves, see [20].

Much of the properties of the KdV equation follow from the interplay between advection and dispersion. One can see that the quantum effects can even invert the sign of dispersion for large H . However, this sign is immaterial since we can apply the transform $t \rightarrow -t, x \rightarrow x, n_i^{(1)} \rightarrow -n_i^{(1)}$. This implies that for $H > 2$, the localized solutions (bright solitons) with $n_i^{(1)} > 0$ of the original equation correspond also to localized solutions, but with inverted polarization ($n_i^{(1)} < 0$, dark solitons) and propagating backward in time. But when $H = 2$, the dispersive term vanishes, which eventually yields the formation of a shock in the Burger's equation. For details of the solitons, one may refer to [11].

When $H \neq 2$, we have the following existence theorem [18, 19].

Theorem 2.1. *Let $H \neq 2$ and $\tilde{s}_1 \geq 2$ be a sufficiently large integer. Then for any given initial data $n_{i0}^{(1)} \in H^{\tilde{s}_1}(\mathbb{R})$, there exists $\tau_* > 0$ such that the initial value problem (2.7) has a unique solution $n_i^{(1)} \in L^\infty(-\tau_*, \tau_*; H^{\tilde{s}_1}(\mathbb{R}))$. Furthermore, by using the conservation laws of the KdV equation, we can extend the solution to any time interval $[-\tau, \tau]$.*

There is also an existence theorem for $H = 2$, see [25, 30].

Theorem 2.2. *Let $H = 2$ and $\tilde{s}_2 \geq 2$ be a sufficiently large integer. Then for any given initial data $n_{i0}^{(1)} \in H^{\tilde{s}_2}(\mathbb{R})$, there exists $\tilde{\tau}_* > 0$ such that the initial value problem (2.7) with $H = 2$ has a unique solution $n_i^{(1)} \in L^\infty(0, \tilde{\tau}_*; H^{\tilde{s}_2}(\mathbb{R}))$ with initial data $n_{i0}^{(1)}$.*

To find out the equation satisfied by $(n_i^{(2)}, n_e^{(2)}, u_i^{(2)})$ assuming $(n_i^{(1)}, n_e^{(1)}, u_i^{(1)})$ is known form (2.5) and (2.7), we express $(n_i^{(2)}, n_e^{(2)}, u_i^{(2)})$ in terms of $(n_i^{(1)}, n_e^{(1)}, u_i^{(1)})$ from (2.6),

$$(\mathcal{L}_2) : \begin{cases} n_e^{(2)} = n_i^{(2)} + \partial_x^2 n_e^{(1)}, & (2.8a) \\ u_i^{(2)} = n_i^{(2)} + g^{(1)}, g^{(1)} = -\int_0^x \mathbf{g}^{(1)}(t, \xi) d\xi, \\ \mathbf{g}^{(1)} = -\partial_t u_i^{(1)} + \partial_\xi (n_i^{(1)} u_i^{(1)}), & (2.8b) \end{cases}$$

which makes (2.6) valid. Thus only $n_i^{(2)}$ needs to be determined.

At the order $O(\epsilon^3)$, we obtain

$$(\mathcal{S}_2) \begin{cases} \partial_t n_i^{(2)} - \partial_x n_i^{(3)} + \partial_x u_i^{(3)} + \partial_x (n_i^{(1)} u_i^{(2)} + n_i^{(2)} u_i^{(1)}) = 0, & (2.9a) \\ \partial_t u_i^{(2)} - \partial_x u_i^{(3)} + \partial_x (u_i^{(1)} u_i^{(2)}) = -\partial_x n_e^{(3)} - \partial_x (n_e^{(1)} n_e^{(2)}) \\ \quad + \frac{H^2}{4} (\partial_x^3 n_e^{(2)} - 2\partial_x n_e^{(1)} \partial_x^2 n_e^{(1)} - n_e^{(1)} \partial_x^3 n_e^{(1)}), & (2.9b) \\ \partial_x^2 n_e^{(2)} + n_e^{(1)} \partial_x^2 n_e^{(1)} + (\partial_x n_e^{(1)})^2 - \frac{H^2}{4} \partial_x^4 n_e^{(1)} = n_e^{(3)} - n_i^{(3)}. & (2.9c) \end{cases}$$

Differentiating (2.9c) with respect to x , and then adding the resultant and (2.9b) to (2.9a) together, we deduce that $n_i^{(2)}$ satisfies the linearized inhomogeneous quantum KdV equation

$$\partial_t n_i^{(2)} + 2\partial_x(n_i^{(1)} n_i^{(2)}) + \frac{1}{2}(1 - \frac{H^2}{4})\partial_x^3 n_i^{(2)} = G^{(1)}, \quad (2.10)$$

where we have used (2.8) and $G^{(1)}$ depending on only $n_i^{(1)}$. Again, the system (2.10) and (2.8) for $(n_i^{(2)}, n_e^{(2)}, u_i^{(2)})$ are self contained, which do not depend on $(n_i^{(j)}, n_e^{(j)}, u_i^{(j)})$ for $j \geq 3$.

Inductively, at the order $O(\epsilon^k)$, we obtain a system (\mathcal{S}_{k-1}) for $(n_i^{(k-1)}, n_e^{(k-1)}, u_i^{(k-1)})$, from which we obtain

$$(\mathcal{L}_k) : \quad n_e^{(k)} = n_i^{(k)} + h^{(k-1)}, \quad u_i^{(k)} = n_i^{(k)} + g^{(k-1)}, \quad (2.11)$$

where $h^{(k-1)}$ and $g^{(k-1)}$ depend only on $(n_e^{(j)})$ for $1 \leq j \leq k-1$. Thus we need only to determine $n_i^{(k)}$. At the order $O(\epsilon^{k+1})$, we obtain a system (\mathcal{S}_k) for $(n_i^{(k)}, n_e^{(k)}, u_i^{(k)})$, from which we obtain the linearized inhomogeneous KdV equation for $n_i^{(k)}$,

$$\partial_t n_i^{(k)} + 2\partial_x(n_i^{(1)} n_i^{(k)}) + \frac{1}{2}(1 - \frac{H^2}{4})\partial_x^3 n_i^{(k)} = G^{(k-1)}, \quad k \geq 3, \quad (2.12)$$

where $G^{(k-1)}$ depends only on $n_i^{(1)}, n_i^{(2)}, \dots, n_i^{(k-1)}$, which are “known” from the first $(k-1)^{th}$ steps. Also, it is important to note that the system (2.11) and (2.12) for $n_i^{(k)}, n_e^{(k)}, u_i^{(k)}$ are self contained, which do not depend on $(n_i^{(j)}, n_e^{(j)}, u_i^{(j)})$ for $j \geq k+1$.

For the solvability of $(n_i^{(k)}, n_e^{(k)}, u_i^{(k)})$ for $k \geq 2$, we state the following

Theorem 2.3. *Let $k \geq 2$, $\tilde{s}_k \leq \tilde{s}_1 - 3(k-1)$ be sufficiently large integers and $n_{i0}^{(k)} \in H^{\tilde{s}_k}(\mathbb{R})$. Then when $H \neq 2$, the initial value problem (2.12) with initial data $n_{i0}^{(k)}$ has a unique solution $n_i^{(k)} \in L^\infty(-\tau, \tau; H^{\tilde{s}_k}(\mathbb{R}))$ for any $\tau > 0$. When $H = 2$, the initial value problem (2.12) has a unique solution $n_i^{(k)} \in L^\infty(0, \tilde{\tau}_*; H^{\tilde{s}_k}(\mathbb{R}))$, where $\tilde{\tau}_*$ is given in Theorem 2.2.*

The proof of Theorem 2.3 is standard. Based on this theorem, we will assume that these solutions $(n_i^{(k)}, n_e^{(k)}, u_i^{(k)})$ for $1 \leq k \leq 4$ are as smooth as we want. The optimality of \tilde{s}_k will not be addressed in this paper.

2.2. Main result. To show that $n_i^{(1)}$ converges to a solution of the KdV equation as $\epsilon \rightarrow 0$, we must make the above procedure rigorous. Let (n_e, n_i, u_i) be the solution of the scaled system (1.3) of the following expansion

$$\begin{cases} n_i &= 1 + \epsilon n_i^{(1)} + \epsilon^2 n_i^{(2)} + \epsilon^3 n_i^{(3)} + \epsilon^4 n_i^{(4)} + \epsilon^3 N_i, \\ n_e &= 1 + \epsilon n_e^{(1)} + \epsilon^2 n_e^{(2)} + \epsilon^3 n_e^{(3)} + \epsilon^4 n_e^{(4)} + \epsilon^3 N_e, \\ u_i &= \epsilon u_i^{(1)} + \epsilon^2 u_i^{(2)} + \epsilon^3 u_i^{(3)} + \epsilon^4 u_i^{(4)} + \epsilon^3 U, \end{cases} \quad (2.13)$$

where $(n_i^{(1)}, n_e^{(1)}, u_i^{(1)})$ satisfies (2.4) and (2.5), $(n_i^{(k)}, n_e^{(k)}, u_i^{(k)})$ satisfies (2.11) and (2.12) for $2 \leq k \leq 4$, and (N_i, N_e, U) is the remainder. To simplify the notation slightly, we set

$$\begin{cases} \tilde{n}_i &= n_i^{(1)} + \epsilon n_i^{(2)} + \epsilon^2 n_i^{(3)} + \epsilon^3 n_i^{(4)}, \\ \tilde{n}_e &= n_e^{(1)} + \epsilon n_e^{(2)} + \epsilon^2 n_e^{(3)} + \epsilon^3 n_e^{(4)}, \\ \tilde{u}_i &= u_i^{(1)} + \epsilon u_i^{(2)} + \epsilon^2 u_i^{(3)} + \epsilon^3 u_i^{(4)}. \end{cases} \quad (2.14)$$

After careful computations, we obtain the following remainder system for (N_i, N_e, U) ,

$$\left\{ \begin{aligned} & \partial_t N_i - \frac{1-u_i}{\epsilon} \partial_x N_i + \frac{n_i}{\epsilon} \partial_x U + \partial_x \tilde{n}_i U + \partial_x \tilde{u}_i N_i + \epsilon \mathcal{R}_1 = 0, \quad (2.15a) \\ & \partial_t U - \frac{1-u_i}{\epsilon} \partial_x U + \partial_x \tilde{u}_i U = -\frac{n_e}{\epsilon} \partial_x N_e - \partial_x \tilde{n}_e N_e \\ & \quad + \frac{H^2}{4} \left[\frac{3\epsilon^2 (\partial_x \tilde{n}_e)^2}{n_e^3} \partial_x N_e - \frac{2\epsilon \partial_x^2 \tilde{n}_e}{n_e^2} \partial_x N_e + \frac{3\epsilon^4 \partial_x \tilde{n}_e}{n_e^3} (\partial_x N_e)^2 \right. \\ & \quad - \frac{2\epsilon \partial_x \tilde{n}_e}{n_e^2} \partial_x^2 N_e + \frac{\epsilon^6}{n_e^3} (\partial_x N_e)^3 - \frac{2\epsilon^3}{n_e^2} \partial_x N_e \partial_x^2 N_e + \frac{1}{n_e} \partial_x^3 N_e \\ & \quad \left. + \frac{\epsilon}{n_e^3} (\mathcal{R}_2^3 + \mathcal{R}_2^4) \right] + \epsilon \mathcal{R}_2^{1,2}, \quad (2.15b) \\ & 2\partial_x \tilde{n}_e \partial_x N_e + \epsilon^2 (\partial_x N_e)^2 + \frac{n_e}{\epsilon} \partial_x^2 N_e + \partial_x^2 \tilde{n}_e N_e + \mathcal{R}_3^1 \\ & \quad - \frac{H^2}{4} \left[-\frac{12\epsilon^3 (\partial_x \tilde{n}_e)^3}{n_e^4} \partial_x N_e + \frac{14\epsilon^2 \partial_x \tilde{n}_e \partial_x^2 \tilde{n}_e}{n_e^3} \partial_x N_e - \frac{3\epsilon \partial_x^3 \tilde{n}_e}{n_e^2} \partial_x N_e \right. \\ & \quad - \frac{18\epsilon^5 (\partial_x \tilde{n}_e)^2}{n_e^4} (\partial_x N_e)^2 + \frac{7\epsilon^2 (\partial_x \tilde{n}_e)^2}{n_e^3} \partial_x^2 N_e + \frac{7\epsilon^4 \partial_x^2 \tilde{n}_e}{n_e^3} (\partial_x N_e)^2 \\ & \quad - \frac{4\epsilon \partial_x^2 \tilde{n}_e}{n_e^2} \partial_x^2 N_e - \frac{12\epsilon^7 \partial_x \tilde{n}_e}{n_e^4} (\partial_x N_e)^3 + \frac{14\epsilon^4 \partial_x \tilde{n}_e}{n_e^3} \partial_x N_e \partial_x^2 N_e \\ & \quad - \frac{3\epsilon \partial_x \tilde{n}_e}{n_e^2} \partial_x^3 N_e - \frac{3\epsilon^9}{n_e^4} (\partial_x N_e)^4 + \frac{7\epsilon^6}{n_e^3} (\partial_x N_e)^2 \partial_x^2 N_e - \frac{2\epsilon^3}{n_e^2} (\partial_x^2 N_e)^2 \\ & \quad \left. - \frac{3\epsilon^3}{n_e^2} \partial_x N_e \partial_x^3 N_e + \frac{\partial_x^4 N_e}{n_e} + \frac{\mathcal{R}_3^2 + \mathcal{R}_3^3}{n_e^4} \right] = \frac{N_e - N_i}{\epsilon^2}, \quad (2.15c) \end{aligned} \right.$$

where

$$\left\{ \begin{aligned} & \mathcal{R}_1 = \partial_x (n_i^{(1)} u_i^{(4)}) + \partial_x [n_i^{(2)} (u_i^{(3)} + \epsilon u_i^{(4)})] + \partial_x [n_i^{(3)} (u_i^{(2)} \\ & \quad + \epsilon u_i^{(3)} + \epsilon^2 u_i^{(4)})] + \partial_x (u_i^{(4)} \tilde{u}_i) - \epsilon \partial_t n_i^{(4)}, \\ & \mathcal{R}_2^1 = -\partial_t u_i^{(4)} + u_i^{(1)} \partial_x u_i^{(4)} + u_i^{(2)} (\partial_x u_i^{(3)} + \epsilon \partial_x u_i^{(4)}) \\ & \quad + u_i^{(3)} (\partial_x u_i^{(2)} + \epsilon \partial_x u_i^{(3)} + \epsilon^2 u_i^{(4)}) + u_i^{(4)} \partial_x \tilde{u}_i, \\ & \mathcal{R}_2^2 = n_e^{(1)} \partial_x n_e^{(4)} + n_e^{(2)} (\partial_x n_e^{(3)} + \epsilon \partial_x n_e^{(4)}) \\ & \quad + n_e^{(3)} (\partial_x n_e^{(2)} + \epsilon \partial_x n_e^{(3)} + \epsilon^2 n_e^{(4)}) + n_e^{(4)} \tilde{n}_e, \\ & \mathcal{R}_2^3 = \mathcal{R}_2^3(n_e^{(1)}, n_e^{(2)}, n_e^{(3)}, n_e^{(4)}), \\ & \mathcal{R}_2^4 = \mathcal{R}_2^4(\epsilon N_e), \\ & \mathcal{R}_3^1 = \mathcal{R}_3^1(n_e^{(1)}, n_e^{(2)}, n_e^{(3)}, n_e^{(4)}), \\ & \mathcal{R}_3^2 = \mathcal{R}_3^2(n_e^{(1)}, n_e^{(2)}, n_e^{(3)}, n_e^{(4)}), \\ & \mathcal{R}_3^3 = \mathcal{R}_3^3(N_e), \end{aligned} \right. \quad (2.16)$$

where \mathcal{R}_2^4 and \mathcal{R}_3^3 are smooth functions of N_e , and do not involve any derivatives of N_e . The mathematical key difficulty is to derive uniform in ϵ estimates for the remainder (N_e, N_i, U) .

For convenient usage, we give the following

Lemma 2.4. *For $\alpha = 0, 1, \dots$ integers, there exists some constant $C = C(\|n_e^{(i)}\|_{H^{\bar{s}_i}})$ such that*

$$\|\mathcal{R}_1, \mathcal{R}_2^{1,2,3}, \mathcal{R}_3^{1,2}\|_{H^\alpha} \leq C, \quad \alpha = 0, 1, \dots, \quad (2.17)$$

$$\begin{aligned} \|\mathcal{R}_2^4\|_{H^\alpha} &\leq C\epsilon \|N_e\|_{H^\alpha}, \quad \alpha = 0, 1, \dots, \\ \|\mathcal{R}_3^3\|_{H^\alpha} &\leq C\|N_e\|_{H^\alpha}, \quad \alpha = 0, 1, \dots, \end{aligned} \quad (2.18)$$

and

$$\begin{aligned}\|\partial_t R_2^4\|_{H^\alpha} &\leq C\epsilon\|\partial_t N_e\|_{H^\alpha}, \quad \alpha = 0, 1, \dots, \\ \|\partial_t R_3^3\|_{H^\alpha} &\leq C\|\partial_t N_e\|_{H^\alpha}, \quad \alpha = 0, 1, \dots.\end{aligned}\tag{2.19}$$

Recalling the fact that H^1 is an algebra, the estimate for Lemma 2.4 is straightforward. The details are hence omitted. Our main result of this paper is the following

Theorem 2.5. *Let \tilde{s}_i be sufficiently large and $(n_i^{(1)}, n_e^{(1)}, u_i^{(1)}) \in H^{\tilde{s}_1}$ be a solution constructed in Theorem 2.1 for the quantum KdV equation with initial data $(n_{i0}, n_{e0}, u_{i0}) \in H^{\tilde{s}_1}$ satisfying (2.5). Let $(n_i^{(j)}, n_e^{(j)}, u_i^{(j)}) \in H^{\tilde{s}_j}$ ($j=2,3,4$) be solutions of (2.11) and (2.12) constructed in Theorem 2.3 with initial data $(n_{i0}^j, n_{e0}^j, u_{i0}^j) \in H^{\tilde{s}_j}$ satisfying (2.11). Let (N_{i0}, N_{e0}, U_0) satisfy (2.15) and assume*

$$\begin{aligned}n_{i0} &= 1 + \epsilon n_{i0}^{(1)} + \epsilon^2 n_{i0}^{(2)} + \epsilon^3 n_{i0}^{(3)} + \epsilon^4 n_{i0}^{(4)} + \epsilon^3 N_{i0}, \\ n_{e0} &= 1 + \epsilon n_{e0}^{(1)} + \epsilon^2 n_{e0}^{(2)} + \epsilon^3 n_{e0}^{(3)} + \epsilon^4 n_{e0}^{(4)} + \epsilon^3 N_{e0}, \\ u_{i0} &= \epsilon u_{i0}^{(1)} + \epsilon^2 u_{i0}^{(2)} + \epsilon^3 u_{i0}^{(3)} + \epsilon^4 u_{i0}^{(4)} + \epsilon^3 U_0.\end{aligned}$$

Then for any $\tau > 0$, there exists $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$, the solution of the EP system (2.2) with initial data (n_{i0}, n_{e0}, u_{i0}) can be expressed as

$$\begin{aligned}n_i &= 1 + \epsilon n_i^{(1)} + \epsilon^2 n_i^{(2)} + \epsilon^3 n_i^{(3)} + \epsilon^4 n_i^{(4)} + \epsilon^3 N_i, \\ n_e &= 1 + \epsilon n_e^{(1)} + \epsilon^2 n_e^{(2)} + \epsilon^3 n_e^{(3)} + \epsilon^4 n_e^{(4)} + \epsilon^3 N_e, \\ u_i &= \epsilon u_i^{(1)} + \epsilon^2 u_i^{(2)} + \epsilon^3 u_i^{(3)} + \epsilon^4 u_i^{(4)} + \epsilon^3 U,\end{aligned}$$

such that for all $0 < \epsilon < \epsilon_0$,

$$\begin{aligned}\sup_{[0, \tau]} \left\{ \|(N_i, N_e, U)\|_{H^2}^2 + \epsilon \|(\partial_x^3 N_e, \partial_x^3 U)\|_{L^2}^2 + \epsilon^2 \|\partial_x^4 N_e\|_{L^2}^2 + \epsilon^3 \|\partial_x^5 N_e\|_{L^2}^2 \right. \\ \left. + \epsilon^4 \|\partial_x^6 N_e\|_{L^2}^2 \right\} \leq C_\tau \left(1 + \|(N_{i0}, N_{e0}, U_0)\|_{H^2}^2 + \epsilon \|(\partial_x^3 N_{e0}, \partial_x^3 U_0)\|_{L^2}^2 \right. \\ \left. + \epsilon^2 \|\partial_x^4 N_{e0}\|_{L^2}^2 + \epsilon^3 \|\partial_x^5 N_{e0}\|_{L^2}^2 + \epsilon^4 \|\partial_x^6 N_{e0}\|_{L^2}^2 \right).\end{aligned}\tag{2.20}$$

From (2.20), we see that the H^2 -norm of the remainder (N_i, N_e, U) is bounded uniformly in ϵ . Note also the Gardner-Morikawa transform (2.1), we see that

$$\sup_{[0, \epsilon^{-3/2}\tau]} \left\| \begin{pmatrix} (n_i - 1)/\epsilon \\ (n_e - 1)/\epsilon \\ u_i/\epsilon \end{pmatrix} - KdV \right\|_{H^2} \leq C\epsilon,\tag{2.21}$$

for some $C > 0$ independent of $\epsilon > 0$. Here ‘KdV’ is the equation satisfied by the first approximation $(n_i^{(1)}, n_e^{(1)}, u_i^{(1)})$.

The basic plan is to first estimate some uniform bound for (N_e, U) and then recover the estimate for N_i from the estimate of N_e by the equation (2.15). We want to apply the Gronwall lemma to complete the proof. To state clearly, we first introduce

$$\|(N_e, U)\|_\epsilon^2 = \|(N_e, U)\|_{H^2}^2 + \epsilon \|(\partial_x^3 N_e, \partial_x^3 U)\|_{L^2}^2 + \epsilon^2 \|\partial_x^4 N_e\|_{L^2}^2 + \epsilon^3 \|\partial_x^5 N_e\|_{L^2}^2 + \epsilon^4 \|\partial_x^6 N_e\|_{L^2}^2.\tag{2.22}$$

As we will see, the zeroth order, the first order to the second order estimates for (N_e, U) and the third order estimates for $\epsilon(N_e, U)$ all can be controlled in terms of $\|(N_e, U)\|_\epsilon^2$.

For convenience, we introduce the following lemma

Lemma 2.6 (Commutator Estimate). *Let $m \geq 1$ be an integer, and then the commutator which is defined by the following*

$$[\nabla^m, f]g := \nabla^m(fg) - f\nabla^m g, \quad (2.23)$$

can be bounded by

$$\|[\nabla^m, f]g\|_{L^p} \leq \|\nabla f\|_{L^{p_1}} \|\nabla^{m-1} g\|_{L^{p_2}} + \|\nabla^m f\|_{L^{p_3}} \|g\|_{L^{p_4}}, \quad (2.24)$$

where $p, p_2, p_3 \in (1, \infty)$ and

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Proof. The proof can be found in [1, 16–19], for example. \square

3. UNIFORM ENERGY ESTIMATES

In this section, we give the energy estimates uniformly in ϵ for the remainder (N_e, N_i, U) , which requires a combination of energy method and analysis of the remainder equation (2.15). To simplify the proof slightly, we assume that (2.15) has smooth solutions in $[0, \tau_\epsilon]$ for $\tau_\epsilon > 0$ depending on ϵ . Let \tilde{C} be a constant independent of ϵ , which will be determined later, much larger than the bound $\|(N_e, U)(0)\|_\epsilon^2$ of the initial data. It is classical that there exists $\tau_\epsilon > 0$ such that on $[0, \tau_\epsilon]$,

$$\|N_i\|_{H^2}^2, \quad \|(N_e, U)\|_\epsilon^2 \leq \tilde{C}. \quad (3.1)$$

As a direct corollary, there exists some $\epsilon_1 > 0$ such that n_e and n_i are bounded from above and below, say $\frac{1}{2} < n_i, n_e < \frac{3}{2}$ and u_i is bounded by $|u_i| < \frac{1}{2}$ when $\epsilon < \epsilon_1$. Since $\mathcal{R}_2^4, \mathcal{R}_3^3$ are smooth functions of N_e , there exists some constant $C_1 = C_1(\epsilon\tilde{C})$ for any $\alpha, \beta \geq 0$ such that

$$\left| \partial_{n_e}^{\alpha} \partial_{N_e}^{\beta} (\mathcal{R}_2^4, \mathcal{R}_3^3) \right| \leq C_1 = C_1(\epsilon\tilde{C}),$$

where $C_1(\cdot)$ can be chosen to be nondecreasing in its argument. We will show that for any given $\tau > 0$ there is some $\epsilon_0 > 0$, such that the existence time $\tau_\epsilon > \tau$ for any $0 < \epsilon < \epsilon_0$.

The purpose of this section is to prove Proposition 3.1 and 3.2. Since the proof of Proposition 3.1 will be almost the same to that of Proposition 3.2, we will omit the proof of Proposition 3.1. In Subsection 3.1, we first show three lemmas that will be frequently used later. In Subsection 3.2 and Subsection 3.3, we present and prove the two main propositions, while estimates of some crucial terms are postponed to Subsection 3.4 and Subsection 3.5.

3.1. Basic estimates. We first prove the following Lemma 3.1–3.3, in which we bound N_i and $\partial_t N_e$ in terms of N_e .

Lemma 3.1. *Let (N_i, N_e, U) be a solution to (2.15) and $\alpha \geq 0$ be an integer. There exist some constants $0 < \epsilon_1 < 1$ and $C_1 = C_1(\epsilon\tilde{C})$ such that for every $0 < \epsilon < \epsilon_1$,*

$$\begin{aligned} C_1^{-1} \|\partial_x^\alpha N_i\|^2 &\leq \|\partial_x^\alpha N_e\|^2 + \epsilon \|\partial_x^{\alpha+1} N_e\|^2 + \epsilon^2 \|\partial_x^{\alpha+2} N_e\|^2 \\ &\quad + \epsilon^3 \|\partial_x^{\alpha+3} N_e\|^2 + \epsilon^4 \|\partial_x^{\alpha+4} N_e\|^2 \leq C_1 \|\partial_x^\alpha N_i\|^2. \end{aligned} \quad (3.2)$$

Proof. When $\alpha = 0$, taking inner product of (1.17c) with N_e and integration by parts, we have

$$\begin{aligned}
& \|N_e\|^2 + \epsilon \|\partial_x N_e\|^2 + \frac{\epsilon^2 H^2}{4} \int \frac{1}{n_e} (\partial_x^2 N_e)^2 \\
&= \epsilon^2 \int (\tilde{n}_e + \epsilon^2 N_e) (\partial_x^2 N_e) N_e - \frac{\epsilon^2 H^2}{2} \int (\partial_x \frac{1}{n_e}) \partial_x^2 N_e \partial_x N_e - \frac{\epsilon^2 H^2}{4} \int (\partial_x^2 \frac{1}{n_e}) \partial_x^2 N_e N_e \\
&\quad + 2\epsilon^2 \int \partial_x \tilde{n}_e (\partial_x N_e) N_e + \epsilon^4 \int (\partial_x N_e)^2 N_e + \epsilon^2 \int \partial_x^2 \tilde{n}_e N_e^2 + \epsilon^2 \int \mathcal{R}_3^1 N_e \\
&\quad - \frac{H^2}{4} \left\{ -12\epsilon^5 \int \frac{(\partial_x \tilde{n}_e)^3}{n_e^4} (\partial_x N_e) N_e + 14\epsilon^4 \int \frac{\partial_x \tilde{n}_e \partial_x^2 \tilde{n}_e}{n_e^3} (\partial_x N_e) N_e \right. \\
&\quad - 3\epsilon^3 \int \frac{\partial_x^3 \tilde{n}_e}{n_e^2} (\partial_x N_e) N_e - 18\epsilon^7 \int \frac{(\partial_x \tilde{n}_e)^2}{n_e^4} (\partial_x N_e)^2 N_e + 7\epsilon^4 \int \frac{(\partial_x \tilde{n}_e)^2}{n_e^3} (\partial_x^2 N_e) N_e \\
&\quad + 7\epsilon^6 \int \frac{\partial_x^2 \tilde{n}_e}{n_e^3} (\partial_x N_e)^2 N_e - 4\epsilon^3 \int \frac{\partial_x^2 \tilde{n}_e}{n_e^2} (\partial_x^2 N_e) N_e - 12\epsilon^9 \int \frac{\partial_x \tilde{n}_e}{n_e^4} (\partial_x N_e)^3 N_e \\
&\quad + 14\epsilon^6 \int \frac{\partial_x \tilde{n}_e}{n_e^3} \partial_x N_e (\partial_x^2 N_e) N_e - 3\epsilon^3 \int \frac{\partial_x \tilde{n}_e}{n_e^2} (\partial_x^3 N_e) N_e - 3\epsilon^{11} \int \frac{1}{n_e^4} (\partial_x N_e)^4 N_e \\
&\quad + 7\epsilon^8 \int \frac{1}{n_e^3} (\partial_x N_e)^2 (\partial_x^2 N_e) N_e - 2\epsilon^5 \int \frac{1}{n_e^2} (\partial_x^2 N_e)^2 N_e - 3\epsilon^5 \int \frac{1}{n_e^2} \partial_x N_e (\partial_x^3 N_e) N_e \\
&\quad \left. + \epsilon^2 \int \frac{\mathcal{R}_3^2 + \mathcal{R}_3^3}{n_e^4} N_e \right\} + \int N_e N_i \\
&=: \sum_{i=1}^{23} A_i.
\end{aligned} \tag{3.3}$$

Since $\frac{1}{2} < n_e < \frac{3}{2}$ and H is a fixed constant, there exists a fixed constant C such that

$$\frac{\epsilon^2 H^2}{4} \int \frac{1}{n_e} (\partial_x^2 N_e)^2 \geq C \epsilon^2 \|\partial_x^2 N_e\|^2.$$

Thus the LHS of (3.3) is equal or greater than $C(\|N_e\|^2 + \epsilon \|\partial_x N_e\|^2 + \epsilon^2 \|\partial_x^2 N_e\|^2)$. Next, we estimate the RHS of (3.3). For A_1 , since \tilde{n}_e is known and bounded in L^∞ , there exists some constant C such that

$$\begin{aligned}
A_1 &= \epsilon^2 \int (\tilde{n}_e + \epsilon^2 N_e) (\partial_x^2 N_e) N_e \\
&\leq C(1 + \epsilon^2 \|N_e\|_{L^\infty}) (\epsilon \|N_e\|^2 + \epsilon^3 \|\partial_x^2 N_e\|^2) \\
&\leq C(1 + \epsilon^2 \|N_e\|_{H^1}) (\epsilon \|N_e\|^2 + \epsilon^3 \|\partial_x^2 N_e\|^2) \\
&\leq C(1 + \epsilon^2 \tilde{C}) (\epsilon \|N_e\|^2 + \epsilon^3 \|\partial_x^2 N_e\|^2) \\
&\leq C(\epsilon \|N_e\|^2 + \epsilon^3 \|\partial_x^2 N_e\|^2),
\end{aligned}$$

where we have used Hölder's inequality, Sobolev embedding $H^1 \hookrightarrow L^\infty$, the priori assumption (3.1) and Cauchy inequality.

Note that

$$\left| \partial_x \left(\frac{1}{n_e} \right) \right| \leq C (\epsilon |\partial_x \tilde{n}_e| + \epsilon^3 |\partial_x N_e|), \tag{3.4}$$

and

$$\left| \partial_x^2 \left(\frac{1}{n_e} \right) \right| \leq C(\epsilon + \epsilon^3(|\partial_x N_e| + |\partial_x^2 N_e|) + \epsilon^6 |\partial_x N_e|^2). \quad (3.5)$$

Since $\partial_x \tilde{n}_e$, $\partial_x^2 \tilde{n}_e$ are bounded in L^∞ , similar to A_1 , we have

$$A_{2 \sim 16, 18 \sim 20} \leq C_1(\epsilon \|N_e\|^2 + \epsilon^2 \|\partial_x N_e\|^2 + \epsilon^3 \|\partial_x^2 N_e\|^2).$$

Now we estimate A_{17} . By integration by parts, we obtain

$$A_{17} = -\frac{3\epsilon^3 H^2}{4} \int (\partial_x \frac{\partial_x \tilde{n}_e}{n_e^2}) \partial_x^2 N_e N_e - \frac{3\epsilon^3 H^2}{4} \int \frac{\partial_x \tilde{n}_e}{n_e^2} \partial_x^2 N_e \partial_x N_e.$$

Similar to (3.4), we have

$$\left| \partial_x \left(\frac{\partial_x \tilde{n}_e}{n_e^2} \right) \right| \leq C(1 + \epsilon^3 |\partial_x N_e|). \quad (3.6)$$

Similar to A_1 , by applying Hölder's inequality, Sobolev embedding $H^1 \hookrightarrow L^\infty$, the priori assumption (3.1) and Cauchy inequality again, we have

$$A_{17} \leq C_1(\epsilon \|N_e\|^2 + \epsilon^2 \|\partial_x N_e\|^2 + \epsilon^3 \|\partial_x^2 N_e\|^2).$$

The term A_{21} can be similarly bounded by

$$A_{21} \leq C_1(\epsilon^2 \|\partial_x N_e\|^2 + \epsilon^3 \|\partial_x^2 N_e\|^2).$$

According to the form of \mathcal{R}_3^2 and \mathcal{R}_3^3 in (1.18), by applying Cauchy inequality, we have

$$A_{22} \leq C_1 \|N_e\|^2.$$

By Young inequality, we have

$$\int N_e N_i \leq \delta \|N_e\|^2 + C_\delta \|N_i\|^2,$$

for arbitrary $\delta > 0$. Hence, there exists some $\epsilon_1 > 0$ such that for $0 < \epsilon < \epsilon_1$,

$$\|N_e\|^2 + \epsilon \|\partial_x N_e\|^2 + \epsilon^2 \|\partial_x^2 N_e\|^2 \leq C_1 \|N_i\|^2. \quad (3.7)$$

Taking inner product of (1.17c) with $\epsilon \partial_x^2 N_e$ and $\epsilon^2 \partial_x^4 N_e$, applying Hölder inequality and integration by parts, we have similarly

$$\epsilon \|\partial_x N_e\|^2 + \epsilon^2 \|\partial_x^2 N_e\|^2 + \epsilon^3 \|\partial_x^3 N_e\|^2 \leq C_1 \|N_i\|^2, \quad (3.8)$$

and

$$\epsilon^2 \|\partial_x^2 N_e\|^2 + \epsilon^3 \|\partial_x^3 N_e\|^2 + \epsilon^4 \|\partial_x^4 N_e\|^2 \leq C_1 \|N_i\|^2. \quad (3.9)$$

By the estimates (3.7), (3.8) and (3.9), we obtain

$$\|N_e\|^2 + \epsilon \|\partial_x N_e\|^2 + \epsilon^2 \|\partial_x^2 N_e\|^2 + \epsilon^3 \|\partial_x^3 N_e\|^2 + \epsilon^4 \|\partial_x^4 N_e\|^2 \leq C_1 \|N_i\|^2. \quad (3.10)$$

On the other hand, from the equation (2.15c), there exist some C such that

$$\|N_i\|^2 \leq C_1(\|N_e\|^2 + \epsilon \|\partial_x N_e\|^2 + \epsilon^2 \|\partial_x^2 N_e\|^2 + \epsilon^3 \|\partial_x^3 N_e\|^2 + \epsilon^4 \|\partial_x^4 N_e\|^2). \quad (3.11)$$

Putting (3.7)-(3.11) together, we deduce the inequality for $\alpha = 0$.

For higher order inequalities, we differentiate (2.15c) with ∂_x^α and then take inner product with $\partial_x^\alpha N_e$, $\epsilon \partial_x^{\alpha+2} N_e$ and $\epsilon^2 \partial_x^{\alpha+4} N_e$ separately. The Lemma then follows by the same procedure of the case $\alpha = 0$. \square

Recall $\|(N_e, U)\|_\epsilon$ in (2.22). We remark that only $\|N_i\|_{H^2}$ can be bounded in terms of $\|(N_e, U)\|_\epsilon$ and no higher order derivatives of N_i are allowed in Lemma 3.1. In fact, we only need $0 \leq \alpha \leq 2$ in Lemma 3.1.

Lemma 3.2. *Let (N_i, N_e, U) be a solution to (2.15). There exist some constants C and $C_1 = C_1(\epsilon\tilde{C})$ such that*

$$\begin{aligned} \|\epsilon\partial_t N_i\|^2 &\leq C(\|N_e\|_{H^1}^2 + \|U\|_{H^1}^2 + \epsilon\|\partial_x^2 N_e\|^2 + \epsilon^2\|\partial_x^3 N_e\|^2 \\ &\quad + \epsilon^3\|\partial_x^4 N_e\|^2 + \epsilon^4\|\partial_x^5 N_e\|^2) + C\epsilon, \end{aligned} \quad (3.12)$$

and

$$\begin{aligned} \|\epsilon\partial_{tx} N_i\|^2 &\leq C_1(\|N_e\|_{H^2}^2 + \|U\|_{H^2}^2 + \epsilon\|\partial_x^3 N_e\|^2 + \epsilon^2\|\partial_x^4 N_e\|^2 \\ &\quad + \epsilon^3\|\partial_x^5 N_e\|^2 + \epsilon^4\|\partial_x^6 N_e\|^2) + C\epsilon. \end{aligned} \quad (3.13)$$

In terms of $\|(N_e, U)\|_\epsilon$, we can rewrite (3.12) and (3.13) as

$$\|\epsilon\partial_t N_i\|_{H^1}^2 \leq C_1\|(N_e, U)\|_\epsilon^2 + C\epsilon.$$

Proof. From (2.15a), we have

$$\epsilon\partial_t N_i = (1 - u_i)\partial_t N_i - n_i\partial_x U - \epsilon\partial_x \tilde{n}_i U - \epsilon\partial_x \tilde{u}_i N_i - \epsilon^2 \mathcal{R}_1.$$

Since $\frac{1}{2} < n_i < \frac{3}{2}$ and $|u_i| < \frac{1}{2}$, taking L^2 -norm yields

$$\begin{aligned} \|\epsilon\partial_t N_i\|^2 &\leq \|(1 - u_i)\partial_t N_i\|^2 + \|n_i\partial_x U\|^2 + \epsilon^2\|\partial_x \tilde{u}_i N_i\|^2 + \epsilon^2\|\partial_x \tilde{n}_i U\|^2 + \epsilon^4\|\mathcal{R}_1\|^2 \\ &\leq C(\|\partial_x N_i\|^2 + \|\partial_x U\|^2) + C\epsilon^2(\epsilon^2 + \|N_i\|^2 + \|U\|^2). \end{aligned}$$

Applying Lemma 3.1 with $\alpha = 1$, we deduce (3.12). To prove (3.13), we take ∂_x of (2.15a) to obtain

$$\|\epsilon\partial_{tx} N_i\|^2 \leq C(\|U\|_{H^2}^2 + \|N_i\|_{H^2}^2) + C\epsilon^6 \int |\partial_x N_i|^2 |\partial_x U|^2 + C\epsilon^4.$$

We note that

$$C\epsilon^6\|\partial_x U\|_{L^\infty}^2\|\partial_x N_i\|^2 \leq C\epsilon^6\|U\|_{H^2}^2\|N_i\|_{H^1}^2 \leq C(\epsilon\tilde{C})\|U\|_{H^2}^2.$$

Applying Lemma 3.1 with $\alpha = 2$, we deduce (3.13). The Lemma then follows from Lemma 3.1. \square

Lemma 3.3. *Let (N_i, N_e, U) be a solution to (2.15) and $\alpha \geq 0$ be an integer. There exist some constants $C_1 = C_1(\epsilon\tilde{C})$ and $\epsilon_1 > 0$ such that for every $0 < \epsilon < \epsilon_1$,*

$$\begin{aligned} \epsilon^4\|\partial_t \partial_x^{\alpha+4} N_e\|^2 + \epsilon^3\|\partial_t \partial_x^{\alpha+3} N_e\|^2 + \epsilon^2\|\partial_t \partial_x^{\alpha+2} N_e\|^2 \\ + \epsilon\|\partial_t \partial_x^{\alpha+1} N_e\|^2 + \|\partial_t \partial_x^\alpha N_e\|^2 \leq C\|\partial_t \partial_x^\alpha N_i\|^2 + C_1. \end{aligned} \quad (3.14)$$

Proof. The proof is similar to that of Lemma 3.1. When $\alpha = 0$, by first taking ∂_t of (2.15c) and then taking inner product with $\partial_t N_e$ and integration by parts, we have

$$\begin{aligned}
& \|\partial_t N_e\|^2 + \epsilon \int n_e (\partial_{tx} N_e)^2 + \frac{\epsilon^2 H^2}{4} \int \frac{1}{n_e} (\partial_t \partial_x^2 N_e)^2 \\
&= -\epsilon \int \partial_x n_e \partial_{tx} N_e \partial_t N_e + \epsilon \int \partial_t n_e \partial_x^2 N_e \partial_t N_e - \frac{\epsilon^2 H^2}{2} \int (\partial_x \frac{1}{n_e}) \partial_t \partial_x^2 N_e \partial_{tx} N_e \\
&\quad - \frac{\epsilon^2 H^2}{4} \int (\partial_x^2 \frac{1}{n_e}) \partial_t \partial_x^2 N_e \partial_t N_e - \frac{\epsilon^2 H^2}{4} \int (\partial_t \frac{1}{n_e}) \partial_x^4 N_e \partial_t N_e \\
&\quad + 2\epsilon^2 \int \partial_t (\partial_x \tilde{n}_e \partial_x N_e) \partial_t N_e + \epsilon^4 \int \partial_t ((\partial_x N_e)^2) \partial_t N_e + \epsilon^2 \int \partial_t (\partial_x^2 \tilde{n}_e N_e) \partial_t N_e \\
&\quad + \epsilon^2 \int \partial_t \mathcal{R}_3^1 \partial_t N_e - \frac{H^2}{4} \left\{ -12\epsilon^5 \int \partial_t \left[\frac{(\partial_x \tilde{n}_e)^3}{n_e^4} \partial_x N_e \right] \partial_t N_e \right. \\
&\quad + 14\epsilon^4 \int \partial_t \left[\frac{\partial_x \tilde{n}_e \partial_x^2 \tilde{n}_e}{n_e^3} \partial_x N_e \right] \partial_t N_e - 3\epsilon^3 \int \partial_t \left[\frac{\partial_x^3 \tilde{n}_e}{n_e^2} \partial_x N_e \right] \partial_t N_e \\
&\quad - 18\epsilon^7 \int \partial_t \left[\frac{(\partial_x \tilde{n}_e)^2}{n_e^4} (\partial_x N_e)^2 \right] \partial_t N_e + 7\epsilon^4 \int \partial_t \left[\frac{(\partial_x \tilde{n}_e)^2}{n_e^3} \partial_x^2 N_e \right] \partial_t N_e \\
&\quad + 7\epsilon^6 \int \partial_t \left[\frac{\partial_x^2 \tilde{n}_e}{n_e^3} (\partial_x N_e)^2 \right] \partial_t N_e - 4\epsilon^3 \int \partial_t \left[\frac{\partial_x^2 \tilde{n}_e}{n_e^2} \partial_x^2 N_e \right] \partial_t N_e \\
&\quad - 12\epsilon^9 \int \partial_t \left[\frac{\partial_x \tilde{n}_e}{n_e^4} (\partial_x N_e)^3 \right] \partial_t N_e + 14\epsilon^6 \int \partial_t \left[\frac{\partial_x \tilde{n}_e}{n_e^3} \partial_x N_e \partial_x^2 N_e \right] \partial_t N_e \\
&\quad - 3\epsilon^3 \int \partial_t \left[\frac{\partial_x \tilde{n}_e}{n_e^2} \partial_x^3 N_e \right] \partial_t N_e - 3\epsilon^{11} \int \partial_t \left[\frac{1}{n_e^4} (\partial_x N_e)^4 \right] \partial_t N_e \\
&\quad + 7\epsilon^8 \int \partial_t \left[\frac{1}{n_e^3} (\partial_x N_e)^2 \partial_x^2 N_e \right] \partial_t N_e - 2\epsilon^5 \int \partial_t \left[\frac{1}{n_e^2} (\partial_x^2 N_e)^2 \right] \partial_t N_e \\
&\quad - 3\epsilon^5 \int \partial_t \left[\frac{1}{n_e^2} \partial_x N_e \partial_x^3 N_e \right] \partial_t N_e + \epsilon^2 \int \partial_t \left[\frac{1}{n_e^4} (\mathcal{R}_3^2 + \mathcal{R}_3^3) \right] \partial_t N_e \Big\} \\
&\quad + \int \partial_t N_i \partial_t N_e \\
&=: \sum_{i=1}^{25} B_i.
\end{aligned} \tag{3.15}$$

Estimate of the LHS of (3.15). Since $\frac{1}{2} < n_e < \frac{3}{2}$ and H is a fixed constant, there exists a fixed constant C such that

$$\epsilon \int n_e (\partial_{tx} N_e)^2 + \frac{\epsilon^2 H^2}{4} \int \frac{1}{n_e} (\partial_t \partial_x^2 N_e)^2 \geq C(\epsilon \|\partial_{tx} N_e\|^2 + \epsilon^2 \|\partial_t \partial_x^2 N_e\|^2).$$

Thus the LHS of (3.15) is equal or greater than $C(\|\partial_t N_e\|^2 + \epsilon \|\partial_{tx} N_e\|^2 + \epsilon^2 \|\partial_t \partial_x^2 N_e\|^2)$. Next, we estimate the righthand side terms. For B_1 , by applying Hölder's inequality, Cauchy inequality and Sobolev embedding $H^1 \hookrightarrow L^\infty$, we have

$$\begin{aligned}
B_1 &= \epsilon^2 \int (\partial_x \tilde{n}_e + \epsilon^2 \partial_x N_e) \partial_{tx} N_e \partial_t N_e \\
&\leq C\epsilon(1 + \epsilon^2 \|\partial_x N_e\|_{L^\infty})(\epsilon \|\partial_t N_e\|^2 + \epsilon^2 \|\partial_{tx} N_e\|^2) \\
&\leq C(\epsilon \tilde{C})(\epsilon \|\partial_t N_e\|^2 + \epsilon^2 \|\partial_{tx} N_e\|^2) \\
&\leq C_1(\epsilon \|\partial_t N_e\|^2 + \epsilon^2 \|\partial_{tx} N_e\|^2),
\end{aligned}$$

where we have used (3.1). Similarly,

$$B_2 \leq C_1(\epsilon \|\partial_t N_e\|^2 + \epsilon^2 \|\partial_{tx} N_e\|^2) + C_1.$$

By (3.1), Sobolev embedding theorem and Cauchy inequality, we have

$$B_3 + B_4 \leq C_1(\epsilon \|\partial_t N_e\|^2 + \epsilon^2 \|\partial_{tx} N_e\|^2 + \epsilon^3 \|\partial_t \partial_x^2 N_e\|^2),$$

where we have used (3.4) and (3.5).

Estimate of B_5 . Similar to (3.4), we note that

$$|\partial_t \frac{1}{n_e}| \leq C(\epsilon |\partial_t \tilde{n}_e| + \epsilon^3 |\partial_t N_e|). \quad (3.16)$$

Therefore, we have

$$\begin{aligned} B_5 &\leq \epsilon^2 \|\partial_x^4 N_e\| (\epsilon \|\partial_t \tilde{n}_e\|_{L^\infty} + \epsilon^3 \|\partial_t N_e\|_{L^\infty}) \|\partial_t N_e\| \\ &\leq \epsilon^2 \tilde{C}(\epsilon C + \epsilon^3 \|\partial_t N_e\|_{H^1}) \|\partial_t N_e\| \\ &\leq C_1(\epsilon \|\partial_t N_e\|^2 + \epsilon^2 \|\partial_{tx} N_e\|^2) + C_1. \end{aligned}$$

Estimate of B_6 . By direct computation, we have

$$\partial_t(\partial_x \tilde{n}_e \partial_x N_e) = \partial_{tx} \tilde{n}_e \partial_x N_e + \partial_x \tilde{n}_e \partial_{tx} N_e,$$

which yields that

$$\|\partial_t(\partial_x \tilde{n}_e \partial_x N_e)\| \leq C(\|\partial_x N_e\| + \|\partial_{tx} N_e\|),$$

where C is a fixed constant. By applying Hölder inequality and Young inequality, we have

$$B_6 \leq C_1(\epsilon \|\partial_t N_e\|^2 + \epsilon^2 \|\partial_{tx} N_e\|^2) + C_1.$$

B_8 is similar to B_6 .

Estimate of B_7 . We note that

$$\|\partial_t[(\partial_x N_e)^2]\| \leq C(\|\partial_x N_e\|_{L^\infty} \|\partial_{tx} N_e\|).$$

Thus, we have

$$B_7 \leq C_1(\epsilon \|\partial_t N_e\|^2 + \epsilon^2 \|\partial_{tx} N_e\|^2),$$

thanks to Hölder inequality, Cauchy inequality and Sobolev embedding $H^1 \hookrightarrow L^\infty$ and (3.1).

Estimate of B_9 . Since \mathcal{R}_3^1 is known, thus by Cauchy inequality, we have

$$B_9 \leq C_1 \epsilon \|\partial_t N_e\|^2 + C_1.$$

Estimate of B_{20} . By direct computation, we have

$$\partial_t \left[\frac{1}{n_e^4} (\partial_x N_e)^4 \right] = \partial_t \left(\frac{1}{n_e^4} \right) (\partial_x N_e)^4 + \frac{4}{n_e^4} (\partial_x N_e)^3 \partial_{tx} N_e.$$

Similar to (3.16), we have

$$\left| \partial_t \left(\frac{1}{n_e^4} \right) \right| \leq C(\epsilon + \epsilon^3 |\partial_t N_e|). \quad (3.17)$$

Thus by applying Hölder inequality, Sobolev embedding $H^1 \hookrightarrow L^\infty$ and (3.1) again, we have

$$\begin{aligned} B_{20} &\leq C \epsilon^{11} \|\partial_t \left[\frac{1}{n_e^4} (\partial_x N_e)^4 \right]\| \|\partial_t N_e\| \\ &\leq C \epsilon^{11} \|\partial_x N_e\|_{L^\infty}^4 (\epsilon + \epsilon^3 \|\partial_t N_e\|) \|\partial_t N_e\| + C \epsilon^{11} \|\partial_x N_e\|_{L^\infty}^3 (\|\partial_t N_e\| \|\partial_{tx} N_e\|) \\ &\leq C_1(\epsilon \|\partial_t N_e\|^2 + \epsilon^2 \|\partial_{tx} N_e\|^2) + C_1. \end{aligned}$$

The estimates of $B_{10\sim 13}$, B_{15} and B_{17} are similar to that for B_{20} .

Estimate of B_{21} . By direct computation, we have

$$\partial_t \left[\frac{1}{n_e^3} (\partial_x N_e)^2 \partial_x^2 N_e \right] = \partial_t \left(\frac{1}{n_e^3} \right) (\partial_x N_e)^2 \partial_x^2 N_e + \frac{2}{n_e^3} \partial_x N_e \partial_x^2 N_e \partial_{tx} N_e + \frac{1}{n_e^3} (\partial_x N_e)^2 \partial_t \partial_x^2 N_e.$$

Thus similarly, we have

$$\begin{aligned} B_{21} &\leq C\epsilon^8 \|\partial_t \left[\frac{1}{n_e^3} (\partial_x N_e)^2 \partial_x^2 N_e \right]\| \|\partial_t N_e\| \\ &\leq C\epsilon^8 [\|\partial_x N_e\|_{L^\infty}^2 \|\partial_x^2 N_e\|_{L^\infty} (\epsilon + \epsilon^3 \|\partial_t N_e\|) + \|\partial_x N_e\|_{L^\infty} \|\partial_x^2 N_e\|_{L^\infty} \|\partial_{tx} N_e\| \\ &\quad + \|\partial_x N_e\|_{L^\infty}^2 \|\partial_t \partial_x^2 N_e\|] \|\partial_t N_e\| \\ &\leq C_1 \|N_e\|_{H^2}^2 (1 + \epsilon \|\partial_x^3 N_e\|^2) \epsilon \|\partial_t N_e\|^2 \\ &\quad + C_1 (\|N_e\|_{H^2}^2 + \epsilon \|\partial_x^3 N_e\|^2) (\epsilon \|\partial_t N_e\|^2 + \epsilon^2 \|\partial_{tx} N_e\|^2) \\ &\quad + C_1 \|N_e\|_{H^2}^2 (\epsilon \|\partial_t N_e\|^2 + \epsilon^3 \|\partial_t \partial_x^2 N_e\|^2) \\ &\leq C_1 (\epsilon \|\partial_t N_e\|^2 + \epsilon^2 \|\partial_{tx} N_e\|^2 + \epsilon^3 \|\partial_t \partial_x^2 N_e\|^2) + C_1. \end{aligned}$$

The estimates of B_{14} , B_{16} , B_{18} and B_{19} are similar to that for B_{21} .

Estimate of B_{23} . By direct computation, we have

$$\begin{aligned} \partial_t \left[\frac{1}{n_e^2} \partial_x N_e \partial_x^3 N_e \right] &= \partial_t \left(\frac{1}{n_e^2} \right) \partial_x N_e \partial_x^3 N_e + \frac{1}{n_e^2} \partial_{tx} N_e \partial_x^3 N_e + \frac{1}{n_e^2} \partial_x N_e \partial_t \partial_x^3 N_e \\ &=: G_1 + G_2 + G_3. \end{aligned}$$

Thus B_{23} is divided three terms

$$B_{23} = \frac{3\epsilon^5 H^2}{4} \sum_{i=1}^3 \int G_i \partial_t N_e =: B_{231} + B_{232} + B_{233}.$$

The first two terms B_{231} and B_{232} can be easily estimated by $C_1(\epsilon \|\partial_t N_e\|^2 + \epsilon^2 \|\partial_{tx} N_e\|^2)$. For the last term B_{233} , we integrate by parts and use Hölder inequality, Cauchy inequality and (3.1) again to obtain

$$\begin{aligned} B_{233} &= -\frac{3H^2}{4} \epsilon^5 \int \left((\partial_x \frac{1}{n_e^2}) \partial_x N_e + \frac{1}{n_e^2} \partial_x^2 N_e \right) \partial_t \partial_x^2 N_e \partial_t N_e \\ &\quad - \frac{3H^2}{4} \epsilon^5 \int \frac{1}{n_e^2} \partial_x N_e \partial_t \partial_x^2 N_e \partial_{tx} N_e \\ &\leq C_1 (\epsilon \|\partial_t N_e\|^2 + \epsilon^2 \|\partial_{tx} N_e\|^2 + \epsilon^3 \|\partial_t \partial_x^2 N_e\|^2), \end{aligned}$$

where we also have used (3.4). Thus we have

$$B_{23} \leq C_1 (\epsilon \|\partial_t N_e\|^2 + \epsilon^2 \|\partial_{tx} N_e\|^2 + \epsilon^3 \|\partial_t \partial_x^2 N_e\|^2).$$

B_{22} is similar to B_{23} .

Estimate of B_{24} . Since \mathcal{R}_3^2 is known, by using (2.19) in Lemma 2.4, we have

$$B_{24} \leq C_1 (1 + \epsilon \|\partial_t N_e\|^2).$$

Estimate of B_{25} . Applying Young inequality, we have

$$B_{25} = \int \partial_t N_i \partial_t N_e \leq \gamma \|\partial_t N_e\|^2 + C_\gamma \|\partial_t N_i\|^2,$$

where for arbitrary small $\gamma > 0$. Hence, we have shown that there exists some $\epsilon_1 > 0$ such that for $0 < \epsilon < \epsilon_1$, we have

$$\|\partial_t N_e\|^2 + \epsilon \|\partial_{tx} N_e\|^2 + \epsilon^2 \|\partial_t \partial_x^2 N_e\|^2 \leq C \|\partial_t N_i\|^2 + C_1. \quad (3.18)$$

Similarly, taking ∂_{tx} of (2.15c) and then taking inner product with $\epsilon \partial_{tx} N_e$, we have

$$\epsilon \|\partial_{tx} N_e\|^2 + \epsilon^2 \|\partial_t \partial_x^2 N_e\|^2 + \epsilon^3 \|\partial_t \partial_x^3 N_e\|^2 \leq C_{\alpha_2} \|\partial_t N_i\|^2 + \epsilon \|\partial_t N_e\|^2 + C_1. \quad (3.19)$$

Taking $\partial_t \partial_x^2$ of (2.15c) and then taking inner product with $\epsilon^2 \partial_t \partial_x^2 N_e$, we have

$$\begin{aligned} \epsilon^2 \|\partial_t \partial_x^2 N_e\|^2 + \epsilon^3 \|\partial_t \partial_x^3 N_e\|^2 + \epsilon^4 \|\partial_t \partial_x^4 N_e\|^2 &\leq C_{\alpha_3} \|\partial_t N_i\|^2 + \epsilon^2 \|\partial_{tx} N_e\|^2 \\ &+ \epsilon \|\partial_t N_e\|^2 + C_1. \end{aligned} \quad (3.20)$$

Putting (3.18), (3.19) and (3.20) together, let $C = \max\{C_{\alpha_1}, C_{\alpha_2}, C_{\alpha_3}\}$, we obtain

$$\|\partial_t N_e\|^2 + \epsilon \|\partial_{tx} N_e\|^2 + \epsilon^2 \|\partial_t \partial_x^2 N_e\|^2 + \epsilon^3 \|\partial_t \partial_x^3 N_e\|^2 + \epsilon^4 \|\partial_t \partial_x^4 N_e\|^2 \leq C \|\partial_t N_i\|^2 + C_1.$$

Thus we have proven (3.14) for $\alpha = 0$. The case of $\alpha \geq 1$ can be proved similarly. \square

3.2. Zeroth, first and second order estimates. The zeroth, first and second order estimates can be summarized in the following

Proposition 3.1. *Let (N_i, N_e, U) be a solution to (2.15) and $\gamma = 0, 1, 2$, then*

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\partial_x^\gamma U\|^2 + \frac{1}{2} \frac{d}{dt} \left[\int \frac{n_e}{n_i} (\partial_x^\gamma N_e)^2 + \epsilon \int \frac{n_e^2}{n_i} (\partial_x^{\gamma+1} N_e)^2 + \frac{\epsilon^2 H^2}{4} \int \frac{1}{n_i} (\partial_x^{\gamma+2} N_e)^2 \right] \\ &+ \frac{1}{2} \frac{\epsilon H^2}{4} \frac{d}{dt} \left[\int \frac{(\partial_x^{\gamma+1} N_e)^2}{n_e n_i} + \epsilon \int \frac{1}{n_i} (\partial_x^{\gamma+2} N_e)^2 + \frac{\epsilon^2 H^2}{4} \int \frac{1}{n_e^2 n_i} (\partial_x^{\gamma+3} N_e)^2 \right] \\ &\leq C_1 (1 + \epsilon \| (N_e, U) \|_\epsilon^4) (1 + \| (N_e, U) \|_\epsilon^2). \end{aligned} \quad (3.21)$$

This proposition can be proved after long tedious calculations, which can be done by the same procedure that used in the proof of Proposition 3.2. Hence we omit the details here for simplicity.

3.3. Third order estimates.

Proposition 3.2. *Let (N_i, N_e, U) be a solution to (2.15) then*

$$\begin{aligned} &\frac{\epsilon}{2} \frac{d}{dt} \|\partial_x^3 U\|^2 + \frac{\epsilon}{2} \frac{d}{dt} \left[\int \frac{n_e}{n_i} (\partial_x^3 N_e)^2 + \epsilon \int \frac{n_e^2}{n_i} (\partial_x^4 N_e)^2 + \frac{\epsilon^2 H^2}{4} \int \frac{1}{n_i} (\partial_x^5 N_e)^2 \right] \\ &+ \frac{\epsilon}{2} \frac{\epsilon H^2}{4} \frac{d}{dt} \left[\int \frac{1}{n_e n_i} (\partial_x^4 N_e)^2 + \epsilon \int \frac{1}{n_i} (\partial_x^5 N_e)^2 + \frac{\epsilon^2 H^2}{4} \int \frac{1}{n_e^2 n_i} (\partial_x^6 N_e)^2 \right] \\ &\leq C_1 (1 + \epsilon^2 \| (N_e, U) \|_\epsilon^6) (1 + \| (N_e, U) \|_\epsilon^2). \end{aligned} \quad (3.22)$$

Proof. We take ∂_x^3 of (2.15b) and then take inner product of $\epsilon \partial_x^3 U$. We obtain

$$\begin{aligned}
& \frac{\epsilon}{2} \frac{d}{dt} \|\partial_x^3 U\|^2 - \int \partial_x^3 ((1 - u_i) \partial_x U) \partial_x^3 U + \epsilon \int \partial_x^3 (\partial_x \tilde{u}_i U) \partial_x^3 U \\
&= - \int \partial_x^3 (n_e \partial_x N_e) \partial_x^3 U + \frac{\epsilon H^2}{4} \int \partial_x^3 \left(\frac{\partial_x^3 N_e}{n_e} \right) \partial_x^3 U - \epsilon \int \partial_x^3 (\partial_x \tilde{n}_e N_e) \partial_x^3 U \\
&+ \frac{H^2}{4} \left\{ 3\epsilon^3 \int \partial_x^3 \left(\frac{(\partial_x \tilde{n}_e)^2}{n_e^3} \partial_x N_e \right) \partial_x^3 U - 2\epsilon^2 \int \partial_x^3 \left(\frac{\partial_x^2 \tilde{n}_e}{n_e^2} \partial_x N_e \right) \partial_x^3 U \right. \\
&+ 3\epsilon^5 \int \partial_x^3 \left(\frac{\partial_x \tilde{n}_e}{n_e^3} (\partial_x N_e)^2 \right) \partial_x^3 U - 2\epsilon^2 \int \partial_x^3 \left(\frac{\partial_x \tilde{n}_e}{n_e^2} \partial_x^2 N_e \right) \partial_x^3 U \\
&+ \epsilon^7 \int \partial_x^3 \left(\frac{(\partial_x N_e)^3}{n_e^3} \right) \partial_x^3 U - 2\epsilon^4 \int \partial_x^3 \left(\frac{\partial_x N_e \partial_x^2 N_e}{n_e^2} \right) \partial_x^3 U \\
&\left. + \epsilon^2 \int \partial_x^3 \left(\frac{\mathcal{R}_2^3 + \mathcal{R}_2^4}{n_e^3} \right) \partial_x^3 U \right\} + \epsilon^2 \int (\partial_x^3 \mathcal{R}_2^{1,2}) \partial_x^3 U \\
&=: \sum_{i=1}^{11} F_i.
\end{aligned} \tag{3.23}$$

Estimate of the LHS of (3.23). First, we estimate the second term on the LHS of (3.23). Using commutator notation (2.23) to rewrite it as

$$- \int \partial_x^3 ((1 - u_i) \partial_x U) \partial_x^3 U = - \int ([\partial_x^3, 1 - u_i] \partial_x U + (1 - u_i) \partial_x^4 U) \partial_x^3 U =: M_1 + M_2$$

We first estimate M_1 . By commutator estimate of Lemma 2.6, we have

$$\|[\partial_x^3, 1 - u_i] \partial_x U\| \leq \|\partial_x(1 - u_i)\|_{L^\infty} \|\partial_x^3 U\| + \|\partial_x^3(1 - u_i)\| \|\partial_x U\|_{L^\infty}.$$

Thus by Hölder inequality, Cauchy inequality and Sobolev embedding theorem $H^1 \hookrightarrow L^\infty$, we have

$$\begin{aligned}
|M_1| &\leq \|[\partial_x^3, 1 - u_i] \partial_x U\| \|\partial_x^3 U\| \\
&\leq C(1 + \epsilon^2 \|\partial_x U\|_{L^\infty}^2) (\epsilon \|\partial_x^3 U\|^2) + C\epsilon(1 + \epsilon^2 \|\partial_x^3 U\|^2) (\|\partial_x U\|_{L^\infty}^2 + \epsilon \|\partial_x^3 U\|^2) \\
&\leq C_1(1 + \epsilon^2 (\|U\|_{H^2}^2 + \epsilon \|\partial_x^3 U\|^2)) (\|U\|_{H^2}^2 + \epsilon \|\partial_x^3 U\|^2) \\
&\leq C_1(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) \|(N_e, U)\|_\epsilon^2,
\end{aligned} \tag{3.24}$$

where $\|(N_e, U)\|_\epsilon^2$ is given in (2.22). Next, we estimate M_2 . By integration by parts, we have

$$\begin{aligned}
|M_2| &= \left| \frac{1}{2} \int \partial_x(1 - u_i) (\partial_x^3 U)^2 \right| \\
&\leq C(1 + \epsilon^2 \|\partial_x U\|_{L^\infty}) (\epsilon \|\partial_x^3 U\|^2) \\
&\leq C(1 + \epsilon^2 \|U\|_{H^2}) (\epsilon \|\partial_x^3 U\|^2),
\end{aligned} \tag{3.25}$$

where we have used Sobolev embedding theorem $H^1 \hookrightarrow L^\infty$. In light of (3.24) and (3.25), we find the second term on the LHS of (3.23) can be bounded by $C(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) \|(N_e, U)\|_\epsilon^2$. The third term on the LHS of (3.23) is bilinear in the unknowns and can be bounded by

$$\epsilon \int \partial_x^3 (\partial_x \tilde{u}_i U) \partial_x^3 U \leq C(\|U\|_{H^2}^2 + \epsilon \|\partial_x^3 U\|^2) \leq C \|(N_e, U)\|_\epsilon^2.$$

Next, we estimate of the RHS of (3.23). We first estimate the terms F_i for $3 \leq i \leq 11$.

Estimate of F_3 . Since F_3 is bilinear in the unknowns, it can be bounded by

$$F_3 \leq C(\|N_e\|_{H^2}^2 + \epsilon \|\partial_x^3 N_e\|^2 + \epsilon \|\partial_x^3 U\|^2),$$

where we have used Cauchy inequality.

Estimate of F_8 . Using commutator notation (2.23), we write

$$F_8 = \frac{\epsilon^7 H^2}{4} \int \left\{ [\partial_x^3, \frac{1}{n_e^3}] (\partial_x N_e)^3 + \frac{1}{n_e^3} \partial_x^3 ((\partial_x N_e)^3) \right\} \partial_x^3 U =: F_{81} + F_{82}.$$

By commutator estimates (2.24) in Lemma 2.6, we have

$$\|[\partial_x^3, \frac{1}{n_e^3}] (\partial_x N_e)^3\| \leq \|\partial_x (\frac{1}{n_e^3})\|_{L^\infty} \|\partial_x^2 ((\partial_x N_e)^3)\| + \|\partial_x^3 (\frac{1}{n_e^3})\| \|\partial_x N_e\|_{L^\infty}^3.$$

By direct computation and Sobolev embedding theorem, we note that

$$\|\partial_x (\frac{1}{n_e^3})\|_{L^\infty} \leq C(\epsilon + \epsilon^3 \|\partial_x N_e\|_{L^\infty}) \leq C(\epsilon + \epsilon^3 \|N_e\|_{H^2}), \quad (3.26)$$

and

$$\begin{aligned} |\partial_x^3 (\frac{1}{n_e^3})| &\leq C(\epsilon + \epsilon^3 (|\partial_x N_e| + |\partial_x^2 N_e| + |\partial_x^3 N_e|)) \\ &\quad + \epsilon^6 (|\partial_x N_e|^2 + |\partial_x N_e| |\partial_x^2 N_e| + \epsilon^9 |\partial_x N_e|^3), \end{aligned} \quad (3.27)$$

which yields that

$$\begin{aligned} \|\partial_x^3 (\frac{1}{n_e^3})\| &\leq C(\epsilon + \epsilon^3 (\|\partial_x N_e\| + \|\partial_x^2 N_e\| + \|\partial_x^3 N_e\|)) \\ &\quad + \epsilon^6 (\|\partial_x N_e\|_{L^\infty} \|\partial_x N_e\| + \|\partial_x N_e\|_{L^\infty} \|\partial_x^2 N_e\|) \\ &\quad + \epsilon^9 \|\partial_x N_e\|_{L^\infty}^2 \|\partial_x N_e\|. \end{aligned} \quad (3.28)$$

By direct computation, we have

$$\|\partial_x^2 [(\partial_x N_e)^3]\| \leq C(\|\partial_x^2 N_e\|_{L^\infty}^2 \|\partial_x N_e\| + \|\partial_x N_e\|_{L^\infty}^2 \|\partial_x^3 N_e\|). \quad (3.29)$$

Therefore, by (3.26), (3.28) and (3.29), and using Hölder inequality and Sobolev embedding $H^1 \hookrightarrow L^\infty$, we can obtain

$$F_{81} \leq C_1 (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^6) (1 + \|(N_e, U)\|_\epsilon^2). \quad (3.30)$$

On the other hand, by direct computation, we have

$$\begin{aligned} \|\partial_x^3 [(\partial_x N_e)^3]\| &\leq C(\|\partial_x^2 N_e\|_{L^\infty}^2 \|\partial_x^3 N_e\| \\ &\quad + \|\partial_x N_e\|_{L^\infty} \|\partial_x^2 N_e\|_{L^\infty} \|\partial_x^3 N_e\| + \|\partial_x N_e\|_{L^\infty}^2 \|\partial_x^4 N_e\|). \end{aligned} \quad (3.31)$$

By applying Hölder inequality and Sobolev embedding theorem again, we have

$$F_{82} \leq C_1 (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) \|(N_e, U)\|_\epsilon^2. \quad (3.32)$$

Adding the estimates (3.30) and (3.32), we have

$$F_8 \leq C_1 (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^5) \|(N_e, U)\|_\epsilon^2.$$

The estimates of $F_4 \sim F_6$ are similar to F_8 and can be bounded by

$$F_{4 \sim 6} \leq C_1 (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^4) \|(N_e, U)\|_\epsilon^2.$$

Estimate of F_9 . Using commutator notation (2.23), we have

$$F_9 = \frac{\epsilon^2 H^2}{2} \int \left\{ [\partial_x^3, \frac{1}{n_e^2}] \partial_x N_e \partial_x^2 N_e + \frac{1}{n_e^2} \partial_x^3 (\partial_x N_e \partial_x^2 N_e) \right\} \partial_x^3 U =: F_{91} + F_{92}.$$

By commutator estimates (2.24) of Lemma 2.6, we have

$$\|[\partial_x^3, \frac{1}{n_e^2}] \partial_x N_e \partial_x^2 N_e\| \leq \|\partial_x (\frac{1}{n_e^2})\|_{L^\infty} \|\partial_x^2 (\partial_x N_e \partial_x^2 N_e)\| + \|\partial_x^3 (\frac{1}{n_e^2})\| \|\partial_x N_e \partial_x^2 N_e\|_{L^\infty}.$$

By direct computation, we note that $\|\partial_x(\frac{1}{n_e^2})\|_{L^\infty}$, $|\partial_x^3(\frac{1}{n_e^2})|$ and $\|\partial_x^3(\frac{1}{n_e^2})\|$ have similar estimates to (3.26), (3.27) and (3.28). Hence we have

$$\|\partial_x^2(\partial_x N_e \partial_x^2 N_e)\| \leq C(\|\partial_x N_e\|_{L^\infty} \|\partial_x^4 N_e\| + \|\partial_x^2 N_e\|_{L^\infty} \|\partial_x^3 N_e\|). \quad (3.33)$$

Therefore, by Hölder inequality and Sobolev embedding theorem, we obtain

$$F_{91} \leq C_1(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^4) \|(N_e, U)\|_\epsilon^2, \quad (3.34)$$

where we have used (3.26), (3.27) and (3.28). On the other hand,

$$\|\partial_x^3(\partial_x N_e \partial_x^2 N_e)\| \leq C(\|\partial_x N_e\|_{L^\infty} \|\partial_x^5 N_e\| + \|\partial_x^2 N_e\|_{L^\infty} \|\partial_x^4 N_e\| + \|\partial_x^3 N_e\|_{L^\infty} \|\partial_x^3 N_e\|).$$

Therefore, by applying Hölder inequality again, we have

$$F_{92} \leq C_1(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) \|(N_e, U)\|_\epsilon^2. \quad (3.35)$$

Adding the estimates (3.34) and (3.35), we have

$$F_9 \leq C_1(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^4) \|(N_e, U)\|_\epsilon^2.$$

F_7 is similar to F_9 . From equation (2.17) and (2.18) in Lemma 2.4, we can obtain

$$F_{10} \leq C_1(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) \|(N_e, U)\|_\epsilon^2,$$

$$F_{11} \leq C_1(1 + \epsilon \|(N_e, U)\|).$$

Estimate of $F_1 + F_2$. By direct computation, we have

$$\begin{aligned} F_1 + F_2 &= \int \left\{ \partial_x^2(n_e \partial_x N_e) - \frac{\epsilon H^2}{4} \partial_x^2 \left(\frac{\partial_x^3 N_e}{n_e} \right) \right\} \partial_x^4 U \\ &= \int \left(n_e \partial_x^3 N_e - \frac{\epsilon H^2}{4} \frac{\partial_x^5 N_e}{n_e} \right) \partial_x^4 U + \int \left(\sum_{\alpha=1}^2 C_2^\alpha \partial_x^\alpha n_e \partial_x^{3-\alpha} N_e \right) \partial_x^4 U \\ &\quad - \frac{\epsilon H^2}{4} \int \left(\sum_{\beta=1}^2 C_2^\beta \partial_x^\beta \left(\frac{1}{n_e} \right) \partial_x^{5-\beta} N_e \right) \partial_x^4 U \\ &=: \sum_{i=1}^3 K_i. \end{aligned}$$

Estimates of K_2 and K_3 . By integration by parts, we have

$$K_2 = - \int \left(\sum_{\alpha=1}^2 C_2^\alpha \partial_x^{\alpha+1} n_e \partial_x^{3-\alpha} N_e + \sum_{\alpha=1}^2 C_2^\alpha \partial_x^\alpha n_e \partial_x^{4-\alpha} N_e \right) \partial_x^3 U.$$

$$K_3 = \frac{\epsilon H^2}{4} \int \left(\sum_{\beta=1}^2 C_2^\beta \partial_x^{\beta+1} \left(\frac{1}{n_e} \right) \partial_x^{5-\beta} N_e + \sum_{\beta=1}^2 C_2^\beta \partial_x^\beta \frac{1}{n_e} \partial_x^{6-\beta} N_e \right) \partial_x^3 U.$$

For K_2 , we have

$$K_2 \leq C_1(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) \|(N_e, U)\|_\epsilon^2.$$

Combining (3.4), (3.5) and (3.27), we obtain

$$K_3 \leq C_1(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^3) \|(N_e, U)\|_\epsilon^2,$$

where we have used Hölder inequality and Sobolev embedding theorem.

Estimates of K_1 . By (2.15a), we have

$$\begin{aligned}\partial_x^4 U &= \frac{1}{n_i} \left\{ \partial_x^3 ((1 - u_i) \partial_x N_i) - \epsilon \partial_t \partial_x^3 N_i \right. \\ &\quad \left. - \sum_{\beta=1}^3 C_3^\beta \partial_x^\beta n_i \partial_x^{4-\beta} U - \epsilon \partial_x^3 (\partial_x \tilde{n}_i U) - \epsilon \partial_x^3 (\partial_x \tilde{u}_i N_i) - \epsilon^2 \partial_x^3 \mathcal{R}_1 \right\} \\ &=: \sum_{i=1}^6 E_i.\end{aligned}$$

Accordingly, K_1 is decomposed into

$$K_1 = \sum_{i=1}^6 \int (n_e \partial_x^3 N_e - \frac{\epsilon H^2}{4} \frac{1}{n_e} \partial_x^5 N_e) E_i =: \sum_{i=1}^6 K_{1i}. \quad (3.36)$$

We first estimate the terms K_{1i} for $3 \leq i \leq 6$ and leave K_{11} and K_{12} in the next two subsections. By Lemma 3.4 and Lemma 3.5 in the next two subsections, we have

$$\begin{aligned}K_{11} &\leq C_1 (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^6) (1 + \|(N_e, U)\|_\epsilon^2), \\ K_{12} &\leq -\frac{\epsilon}{2} \frac{d}{dt} \left[\int \frac{n_e}{n_i} (\partial_x^3 N_e)^2 + \epsilon \int \frac{n_e^2}{n_i} (\partial_x^4 N_e)^2 + \frac{\epsilon^2 H^2}{4} \int \frac{1}{n_i} (\partial_x^5 N_e)^2 \right] \\ &\quad - \frac{\epsilon}{2} \frac{\epsilon H^2}{4} \frac{d}{dt} \left[\int \frac{1}{n_e n_i} (\partial_x^4 N_e)^2 + \epsilon \int \frac{1}{n_i} (\partial_x^5 N_e)^2 + \frac{\epsilon^2 H^2}{4} \int \frac{1}{(n_e)^2 n_i} (\partial_x^6 N_e)^2 \right] \\ &\quad + C_1 (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^6) (1 + \|(N_e, U)\|_\epsilon^2).\end{aligned}$$

Estimate of K_{13} . It can be decomposed that

$$K_{13} = \int \left(\frac{n_e}{n_i} \partial_x^3 N_e - \frac{\epsilon H^2}{4} \frac{1}{n_e n_i} \partial_x^5 N_e \right) \sum_{\beta=1}^3 C_3^\beta \partial_x^\beta n_i \partial_x^{4-\beta} U.$$

When $\beta = 1, 2$, K_{13} can be easily bounded by $C_1 (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) \|(N_e, U)\|_\epsilon^2$ by Hölder inequality, Cauchy inequality and Lemma 3.1. When $\beta = 3$, by integration by parts, we have

$$\begin{aligned}K_{13} &= - \int \left(\partial_x \left(\frac{n_e}{n_i} \right) \partial_x^3 N_e - \frac{\epsilon H^2}{4} \partial_x \left(\frac{1}{n_e n_i} \right) \partial_x^5 N_e \right) \partial_x^2 n_i \partial_x U \\ &\quad - \int \left(\frac{n_e}{n_i} \partial_x^4 N_e - \frac{\epsilon H^2}{4} \frac{1}{n_e n_i} \partial_x^6 N_e \right) \partial_x^2 n_i \partial_x U \\ &\quad - \int \left(\frac{n_e}{n_i} \partial_x^3 N_e - \frac{\epsilon H^2}{4} \frac{1}{n_e n_i} \partial_x^5 N_e \right) \partial_x^2 n_i \partial_x^2 U \\ &=: K_{131} + K_{132} + K_{133}.\end{aligned}$$

By direct computation, we have

$$\left| \partial_x \left(\frac{1}{n_e n_i} \right) \right|, \left| \partial_x \left(\frac{n_e}{n_i} \right) \right| \leq C (\epsilon + \epsilon^3 (|\partial_x N_e| + |\partial_x N_i|)). \quad (3.37)$$

Therefore, by Hölder inequality, Sobolev embedding $H^1 \hookrightarrow L^\infty$ and Lemma 3.1, we have

$$\begin{aligned}K_{131} &\leq C_1 (1 + \epsilon^2 (\|\partial_x N_e\|_{L^\infty}^2 + \|\partial_x N_i\|_{L^\infty}^2 + \|\partial_x U\|_{L^\infty}^2)) (\epsilon \|\partial_x^3 N_e\|^2 + \|\partial_x^2 N_i\|^2 + \epsilon^3 \|\partial_x^5 N_e\|^2) \\ &\leq C_1 (1 + \epsilon^2 (\|N_e\|_{H^2}^2 + \|N_i\|_{H^2}^2 + \|U\|_{H^2}^2)) (\epsilon \|\partial_x^3 N_e\|^2 + \|\partial_x^2 N_i\|^2 + \epsilon^3 \|\partial_x^5 N_e\|^2) \\ &\leq C_1 (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) \|(N_e, U)\|_\epsilon^2.\end{aligned}$$

On the other hand, by Hölder inequality and Lemma 3.1, K_{132} and K_{133} can be bounded by $C_1 (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) \|(N_e, U)\|_\epsilon^2$. Thus K_{13} is bounded by $C_1 (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) \|(N_e, U)\|_\epsilon^2$.

Estimate of K_{14} . By Hölder inequality and Lemma 3.1, K_{14} can be bounded easily by $C_1 \|(N_e, U)\|_\epsilon^2$.

Estimate of K_{15} . By applying integration by parts and (3.37), we have

$$\begin{aligned} K_{15} &= -\epsilon \int \left(\partial_x \left(\frac{n_e}{n_i} \right) \partial_x^3 N_e - \frac{\epsilon H^2}{4} \partial_x \left(\frac{1}{n_e n_i} \right) \partial_x^5 N_e \right) \partial_x^2 (\partial_x \tilde{u}_i N_i) \\ &\quad - \epsilon \int \left(\frac{n_e}{n_i} \partial_x^4 N_e - \frac{\epsilon H^2}{4} \frac{1}{n_e n_i} \partial_x^6 N_e \right) \partial_x^2 (\partial_x \tilde{u}_i N_i) \\ &\leq C_1 (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) \|(N_e, U)\|_\epsilon^2, \end{aligned}$$

where we have used Hölder inequality and the Lemma 3.1.

Estimate of K_{16} . Since \mathcal{R}_1 is known, thus we have

$$K_{16} \leq C_1 (\epsilon \|\partial_x^3 N_e\| + \epsilon^3 \|\partial_x^5 N_e\|).$$

Summarizing all the estimates, we complete the proof of Proposition 3.2. \square

3.4. Estimate of K_{11} . Next we estimate K_{11} in (3.36).

Lemma 3.4 (*Estimate of K_{11}*). *Let (N_i, N_e, U) be a solution to (2.15) then*

$$K_{11} \leq C_1 (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^6) (1 + \|(N_e, U)\|_\epsilon^2).$$

Proof. Recall that in (3.36),

$$\begin{aligned} K_{11} &= \int \left(\frac{n_e}{n_i} \partial_x^3 N_e - \frac{\epsilon H^2}{4} \frac{1}{n_e n_i} \partial_x^5 N_e \right) \partial_x^3 ((1 - u_i) \partial_x N_i) \\ &= \int \left(\frac{n_e}{n_i} \partial_x^3 N_e - \frac{\epsilon H^2}{4} \frac{1}{n_e n_i} \partial_x^5 N_e \right) \sum_{\gamma=0}^3 C_3^\gamma \partial_x^{3-\gamma} (1 - u_i) \partial_x^{\gamma+1} N_i. \end{aligned}$$

When $\gamma = 0, 1$, by Hölder inequality, Sobolev embedding $H^1 \hookrightarrow L^\infty$ and Lemma 3.1,

$$\begin{aligned} K_{11}|_{\gamma=0,1} &\leq C_1 (1 + \epsilon^2 (\epsilon \|\partial_x^3 U\|^2)) (\|N_i\|_{H^2}^2 + \epsilon \|\partial_x^3 N_e\|^2 + \epsilon^3 \|\partial_x^5 N_e\|^2) \\ &\leq C_1 (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) (1 + \|(N_e, U)\|_\epsilon^2). \end{aligned}$$

By integration by parts for $\gamma = 2$ and (3.37), we have

$$\begin{aligned} K_{11}|_{\gamma=2} &= 3 \int \left(\partial_x \left(\frac{n_e}{n_i} \right) \partial_x^3 N_e - \frac{\epsilon H^2}{4} \partial_x \left(\frac{1}{n_e n_i} \right) \partial_x^5 N_e \right) \partial_x u_i \partial_x^2 N_i \\ &\quad + 3 \int \left(\frac{n_e}{n_i} \partial_x^4 N_e - \frac{\epsilon H^2}{4} \frac{1}{n_e n_i} \partial_x^6 N_e \right) \partial_x u_i \partial_x^2 N_i \\ &\quad + 3 \int \left(\frac{n_e}{n_i} \partial_x^3 N_e - \frac{\epsilon H^2}{4} \frac{1}{n_e n_i} \partial_x^5 N_e \right) \partial_x^2 u_i \partial_x^2 N_i \\ &\leq C (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) \|(N_e, U)\|_\epsilon^2, \end{aligned}$$

where we have used Hölder inequality and Lemma 3.1.

In the following we estimate K_{11} for $\gamma = 3$, by (2.15c), we have

$$\begin{aligned}
\partial_x^4 N_i &= \partial_x^4 N_e - \epsilon \partial_x^4 (n_e \partial_x^2 N_e) - 2\epsilon^2 \partial_x^4 (\partial_x \tilde{n}_e \partial_x N_e) - \epsilon^4 \partial_x^4 ((\partial_x N_e)^2) - \epsilon^2 \partial_x^4 (\partial_x^2 \tilde{n}_e N_e) \\
&\quad - \epsilon^2 \partial_x^4 R_3^1 + \frac{H^2}{4} \left[-12\epsilon^5 \partial_x^4 \left(\frac{(\partial_x \tilde{n}_e)^3}{n_e^4} \partial_x N_e \right) + 14\epsilon^4 \partial_x^4 \left(\frac{\partial_x \tilde{n}_e \partial_x^2 \tilde{n}_e}{n_e^3} \partial_x N_e \right) \right. \\
&\quad - 3\epsilon^3 \partial_x^4 \left(\frac{\partial_x^3 \tilde{n}_e}{n_e^2} \partial_x N_e \right) - 18\epsilon^7 \partial_x^4 \left(\frac{(\partial_x \tilde{n}_e)^2}{n_e^4} (\partial_x N_e)^2 \right) + 7\epsilon^4 \partial_x^4 \left(\frac{(\partial_x \tilde{n}_e)^2}{n_e^3} \partial_x^2 N_e \right) \\
&\quad + 7\epsilon^6 \partial_x^4 \left(\frac{\partial_x^2 \tilde{n}_e}{n_e^3} (\partial_x N_e)^2 \right) - 4\epsilon^3 \partial_x^4 \left(\frac{\partial_x^2 \tilde{n}_e}{n_e^2} \partial_x^2 N_e \right) - 12\epsilon^9 \partial_x^4 \left(\frac{\partial_x \tilde{n}_e}{n_e^4} (\partial_x N_e)^3 \right) \\
&\quad + 14\epsilon^6 \partial_x^4 \left(\frac{\partial_x \tilde{n}_e}{n_e^3} \partial_x N_e \partial_x^2 N_e \right) - 3\epsilon^3 \partial_x^4 \left(\frac{\partial_x \tilde{n}_e}{n_e^2} \partial_x^3 N_e \right) - 3\epsilon^{11} \partial_x^4 \left(\frac{1}{n_e^4} (\partial_x N_e)^4 \right) \\
&\quad + 7\epsilon^8 \partial_x^4 \left(\frac{1}{n_e^3} (\partial_x N_e)^2 \partial_x^2 N_e \right) - 2\epsilon^5 \partial_x^4 \left(\frac{1}{n_e^2} (\partial_x^2 N_e)^2 \right) - 3\epsilon^5 \partial_x^4 \left(\frac{1}{n_e^2} \partial_x N_e \partial_x^3 N_e \right) \\
&\quad \left. + \epsilon^2 \partial_x^4 \left(\frac{\partial_x^4 N_e}{n_e} \right) + \epsilon^2 \partial_x^4 \left(\frac{R_3^2 + R_3^3}{n_e^4} \right) \right] \\
&=: \sum_{i=1}^{22} F_i.
\end{aligned}$$

Accordingly, $K_{11}|_{\gamma=3}$ is decomposed into

$$K_{11}|_{\gamma=3} = \sum_{i=1}^{22} \int \left(\frac{n_e}{n_i} \partial_x^3 N_e - \frac{\epsilon H^2}{4} \frac{1}{n_e n_i} \partial_x^5 N_e \right) (1 - u_i) F_i =: \sum_{i=1}^{22} J_i.$$

Estimate of J_1 . By integration by parts, we have

$$J_1 = -\frac{1}{2} \int (\partial_x^3 N_e)^2 \partial_x \left(\frac{n_e(1 - u_i)}{n_i} \right) + \frac{\epsilon H^2}{8} \int (\partial_x^4 N_e)^2 \partial_x \left(\frac{1 - u_i}{n_e n_i} \right).$$

By direct computation, we have

$$\left| \partial_x \frac{1 - u_i}{n_e n_i} \right|, \quad \left| \partial_x \frac{n_e(1 - u_i)}{n_i} \right| \leq C (\epsilon + \epsilon^3 (|\partial_x N_e| + |\partial_x N_i| + |\partial_x U|)). \quad (3.38)$$

Hence by Hölder inequality, Sobolev embedding $H^1 \hookrightarrow L^\infty$ and Lemma 3.1, we have

$$\begin{aligned}
J_1 &\leq C(1 + \epsilon^2 (\|N_e\|_{H^2}^2 + \|N_i\|_{H^2}^2 + \|U\|_{H^2}^2)) (\epsilon \|\partial_x^3 N_e\|^2 + \epsilon^2 \|\partial_x^4 N_e\|^2) \\
&\leq C((1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) \|(N_e, U)\|_\epsilon^2).
\end{aligned}$$

Estimate of J_2 . By integration by parts, we have

$$\begin{aligned}
J_2 &= -\epsilon \int \left[\left(\partial_x \frac{n_e(1 - u_i)}{n_i} \right) \partial_x^3 N_e - \frac{\epsilon H^2}{4} \left(\partial_x \frac{1 - u_i}{n_e n_i} \right) \partial_x^5 N_e \right] \sum_{\alpha=0}^3 C_\alpha^3 \partial_x^\alpha n_e \partial_x^{5-\alpha} N_e \\
&\quad - \epsilon \int \left[\frac{n_e(1 - u_i)}{n_i} \partial_x^4 N_e - \frac{\epsilon H^2}{4} \frac{1 - u_i}{n_e n_i} \partial_x^6 N_e \right] \sum_{\beta=0}^3 C_\beta^3 \partial_x^\beta n_e \partial_x^{5-\beta} N_e \\
&=: J_{21} + J_{22}.
\end{aligned}$$

Using the equation (3.38), and by Hölder inequality and Sobolev embedding theorem,

$$J_{21} \leq C((1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) \|(N_e, U)\|_\epsilon^2).$$

When $\beta = 0, 1, 2$, J_{22} can be easily estimated by $C(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) \|(N_e, U)\|_\epsilon^2$. When $\beta = 3$, by integration by parts, we have

$$J_{22} = \frac{\epsilon}{2} \int \left[\left(\partial_x \frac{n_e^2(1-u_i)}{n_i} \right) (\partial_x^4 N_e)^2 - \frac{\epsilon H^2}{4} \left(\partial_x \frac{1-u_i}{n_i} \right) (\partial_x^5 N_e)^2 \right].$$

Similar to (3.38), we have

$$\begin{aligned} \left| \partial_x \frac{n_e^2(1-u_i)}{n_i} \right| &\leq C (\epsilon + \epsilon^3 (|\partial_x N_e| + |\partial_x N_i| + |\partial_x U|)), \\ \left| \partial_x \frac{1-u_i}{n_i} \right| &\leq C (\epsilon + \epsilon^3 (|\partial_x N_i| + |\partial_x U|)). \end{aligned} \quad (3.39)$$

Therefore, J_{22} can be estimated by $C(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) \|(N_e, U)\|_\epsilon^2$. As a result, J_2 can be estimated by $C(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) \|(N_e, U)\|_\epsilon^2$. $J_3 \sim J_5$ can be also estimated by $C(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) \|(N_e, U)\|_\epsilon^2$.

Estimate of J_{17} . Using commutator notation (2.23), we have

$$\begin{aligned} J_{17} = & -\frac{3\epsilon^{11}H^2}{4} \int \left(\frac{n_e(1-u_i)}{n_i} \partial_x^3 N_e - \frac{\epsilon H^2}{4} \frac{1-u_i}{n_e n_i} \partial_x^5 N_e \right) \\ & \times \left\{ \left[\partial_x^4, \frac{1}{n_e^4} \right] (\partial_x N_e)^4 + \frac{1}{n_e^4} \partial_x^4 (\partial_x N_e)^4 \right\} =: J_{171} + J_{172}. \end{aligned}$$

By commutator estimates (2.24), we have

$$\left\| \left[\partial_x^4, \frac{1}{n_e^4} \right] (\partial_x N_e)^4 \right\| \leq \left\| \partial_x \left(\frac{1}{n_e^4} \right) \right\|_{L^\infty} \|\partial_x^3 (\partial_x N_e)^4\| + \left\| \partial_x^4 \left(\frac{1}{n_e^4} \right) \right\| \|\partial_x N_e\|_{L^\infty}^4.$$

By direct computation, we have

$$\begin{aligned} \left\| \partial_x^4 \left(\frac{1}{n_e^4} \right) \right\| &\leq C \left(1 + \epsilon^3 (\|\partial_x N_e\| + \|\partial_x^2 N_e\| + \|\partial_x^3 N_e\| + \|\partial_x^4 N_e\|) \right. \\ &\quad + \epsilon^6 (\|\partial_x N_e\|_{L^\infty} \|\partial_x N_e\|_{H^2} + \|\partial_x^2 N_e\|_{L^\infty} \|\partial_x^2 N_e\|) \\ &\quad + \epsilon^9 (\|\partial_x N_e\|_{L^\infty}^2 \|\partial_x^2 N_e\| + \|\partial_x N_e\|_{L^\infty}^2 \|\partial_x N_e\|) \\ &\quad \left. + \epsilon^{12} \|\partial_x N_e\| \|\partial_x N_e\|_{L^\infty}^3 \right), \end{aligned} \quad (3.40)$$

and

$$\begin{aligned} \|\partial_x^3 (\partial_x N_e)^4\| &\leq C (\|\partial_x N_e\|_{L^\infty}^2 \|\partial_x^2 N_e\|_{L^\infty} \|\partial_x^3 N_e\| \\ &\quad + \|\partial_x N_e\|_{L^\infty} \|\partial_x^2 N_e\|_{L^\infty}^2 \|\partial_x^2 N_e\| + \|\partial_x N_e\|_{L^\infty}^3 \|\partial_x^4 N_e\|). \end{aligned} \quad (3.41)$$

Therefore, using (3.26), (3.40) and (3.41), by Hölder inequality and Sobolev embedding theorem, we obtain

$$J_{171} \leq C_1 (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^8) \|(N_e, U)\|_\epsilon^2. \quad (3.42)$$

On the other hand, by direct computation, we have

$$\begin{aligned} \|\partial_x^4 (\partial_x N_e)^4\| &\leq C (\|\partial_x^2 N_e\|_{L^\infty}^3 \|\partial_x^2 N_e\| + \|\partial_x N_e\|_{L^\infty} \|\partial_x^2 N_e\|_{L^\infty}^2 \|\partial_x^3 N_e\| \\ &\quad + \|\partial_x N_e\| \|\partial_x^3 N_e\|_{L^\infty}^2 + \|\partial_x N_e\|_{L^\infty}^2 \|\partial_x^2 N_e\|_{L^\infty} \|\partial_x^4 N_e\| \\ &\quad + \|\partial_x N_e\|_{L^\infty}^3 \|\partial_x^5 N_e\|). \end{aligned} \quad (3.43)$$

Therefore, by applying Hölder inequality again, we have

$$J_{172} \leq C_1 (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^4) \|(N_e, U)\|_\epsilon^2. \quad (3.44)$$

Adding the estimates (3.42) and (3.44), we have

$$J_{17} \leq C_1(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^8) \|(N_e, U)\|_\epsilon^2.$$

$J_7 \sim J_{10}$, J_{12} , J_{14} are similar to J_{17} .

Estimate of J_{19} . Using commutator notation (2.23), we have

$$\begin{aligned} J_{19} = & \frac{1}{2} \epsilon^5 H^2 \int \left(\frac{n_e(1-u_i)}{n_i} \partial_x^3 N_e - \frac{\epsilon H^2}{4} \frac{1-u_i}{n_e n_i} \partial_x^5 N_e \right) \\ & \times \left\{ \left[\partial_x^4, \frac{1}{n_e^2} \right] (\partial_x^2 N_e)^2 + \frac{1}{n_e^2} \partial_x^4 ((\partial_x^2 N_e)^2) \right\} =: J_{191} + J_{192}. \end{aligned}$$

By commutator estimates (2.24), we have

$$\| [\partial_x^4, \frac{1}{n_e^2}] (\partial_x^2 N_e)^2 \| \leq \| \partial_x \frac{1}{n_e^2} \|_{L^\infty} \| \partial_x^3 (\partial_x^2 N_e)^2 \| + \| \partial_x^4 \frac{1}{n_e^2} \| \| \partial_x^2 N_e \|_{L^\infty}^2.$$

By direct computation, we have

$$\| \partial_x^3 (\partial_x^2 N_e)^2 \| \leq C (\| \partial_x^3 N_e \|_{L^\infty} \| \partial_x^4 N_e \| + \| \partial_x^2 N_e \|_{L^\infty} \| \partial_x^5 N_e \|). \quad (3.45)$$

Therefore, using (3.26), (3.40) and (3.45), by Hölder inequality and Sobolev embedding theorem, we can obtain

$$J_{191} \leq C_1(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^6) \|(N_e, U)\|_\epsilon^2. \quad (3.46)$$

On the other hand, by direct computation, we have

$$\| \partial_x^4 (\partial_x^2 N_e)^2 \| \leq C (\| \partial_x^4 N_e \|_{L^\infty} \| \partial_x^4 N_e \| + \| \partial_x^3 N_e \|_{L^\infty} \| \partial_x^5 N_e \| + \| \partial_x^2 N_e \|_{L^\infty} \| \partial_x^6 N_e \|). \quad (3.47)$$

Therefore, by applying Hölder inequality again, we have

$$J_{192} \leq C_1(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) \|(N_e, U)\|_\epsilon^2. \quad (3.48)$$

Adding the estimates (3.46) and (3.48), we have

$$J_{19} \leq C_1(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^6) \|(N_e, U)\|_\epsilon^2.$$

J_{11} , J_{13} , J_{15} , J_{18} are similar to J_{19} .

Estimate of J_{21} .

$$J_{21} = \frac{\epsilon^2 H^2}{4} \int \left(\frac{n_e}{n_i} \partial_x^3 N_e - \frac{\epsilon H^2}{4} \frac{1}{n_e n_i} \partial_x^5 N_e \right) (1-u_i) \partial_x^4 \left(\frac{\partial_x^4 N_e}{n_e} \right) =: J_{211} + J_{212}.$$

By integration of parts twice and commutator notation (2.23), we have

$$\begin{aligned} J_{211} = & \frac{\epsilon^2 H^2}{4} \int \left(\frac{n_e(1-u_i)}{n_i} \partial_x^5 N_e + 2 \partial_x \left(\frac{n_e(1-u_i)}{n_i} \right) \partial_x^4 N_e + \partial_x^2 \left(\frac{n_e(1-u_i)}{n_i} \right) \partial_x^3 N_e \right) \\ & \times \left([\partial_x^2, \frac{1}{n_e}] \partial_x^4 N_e + \frac{1}{n_e} \partial_x^6 N_e \right). \end{aligned}$$

By commutator estimates (2.24) of Lemma 2.6, we have

$$\| [\partial_x^2, \frac{1}{n_e}] \partial_x^4 N_e \| \leq \| \partial_x \frac{1}{n_e} \|_{L^\infty} \| \partial_x^5 N_e \| + \| \partial_x^2 \frac{1}{n_e} \| \| \partial_x^4 N_e \|_{L^\infty}.$$

By direct computation, we have

$$\begin{aligned} \|\partial_x^2(\frac{n_e(1-u_i)}{n_i})\| &\leq C(\epsilon + \epsilon^3(\|\partial_x N_e\| + \|\partial_x N_i\| + \|\partial_x U\|) \\ &\quad + \epsilon^6(\|\partial_x N_e\|_{L^\infty}\|\partial_x N_i\| + \|\partial_x N_i\|_{L^\infty}\|\partial_x U\| + \|\partial_x N_e\|_{L^\infty}\|\partial_x U\| \\ &\quad + \|\partial_x N_e\|_{L^\infty}\|\partial_x N_e\| + \|\partial_x N_i\|_{L^\infty}\|\partial_x N_i\| + \|\partial_x U\|_{L^\infty}\|\partial_x U\|)). \end{aligned} \quad (3.49)$$

Therefore, by applying Hölder inequality, Sobolev embedding theorem and Lemma 3.1,

$$J_{211} \leq C_1(1 + \epsilon^2\|(N_e, U)\|_\epsilon^4)\|(N_e, U)\|_\epsilon^2.$$

By integration by parts and commutator notation (2.23), we have

$$\begin{aligned} J_{212} &= \frac{\epsilon^3 H^4}{16} \int (\frac{1-u_i}{n_e n_i} \partial_x^6 N_e + \partial_x(\frac{1-u_i}{n_e n_i}) \partial_x^5 N_e) ([\partial_x^3, \frac{1}{n_e}] \partial_x^4 N_e + \frac{1}{n_e} \partial_x^7 N_e) \\ &=: J_{2121} + J_{2122}. \end{aligned}$$

Similar to J_{211} , using (3.26) and (3.28), Sobolev embedding, Cauchy inequality and Lemma 3.1, we have

$$J_{2121} \leq C_1(1 + \epsilon^2\|(N_e, U)\|_\epsilon^4)\|(N_e, U)\|_\epsilon^2. \quad (3.50)$$

By integration by parts, we have

$$\begin{aligned} J_{2122} &= -\frac{\epsilon^3 H^4}{16} \int \left(\frac{1}{2} \partial_x \left(\frac{1-u_i}{n_e^2 n_i} \right) + \frac{1}{n_e} \partial_x \left(\frac{1}{n_e n_i} \right) \right) (\partial_x^6 N_e)^2 \\ &\quad + \int \partial_x \left(\frac{1}{n_e} \partial_x \left(\frac{1-u_i}{n_e n_i} \right) \right) \partial_x^5 N_e \partial_x^6 N_e. \end{aligned}$$

Note that

$$\begin{aligned} \left\| \partial_x \left(\frac{1}{n_e} \partial_x \left(\frac{1-u_i}{n_e n_i} \right) \right) \right\|_{L^\infty} &\leq C(\epsilon + \epsilon^3(\|\partial_x N_e\|_{L^\infty} + \|\partial_x N_i\|_{L^\infty} + \|\partial_x U\|_{L^\infty}) \\ &\quad + \epsilon^6(\|\partial_x N_e\|_{L^\infty}^2 + \|\partial_x N_i\|_{L^\infty}^2 + \|\partial_x U\|_{L^\infty}^2 \\ &\quad + \|\partial_x N_e \partial_x N_i\|_{L^\infty} + \|\partial_x N_e \partial_x U\|_{L^\infty} + \|\partial_x N_i \partial_x U\|_{L^\infty})). \end{aligned} \quad (3.51)$$

Thus, by (3.37), (3.39), Hölder inequality, Sobolev embedding theorem and Lemma 3.1,

$$J_{2122} \leq C(1 + \epsilon^2\|(N_e, U)\|_\epsilon^2)\|(N_e, U)\|_\epsilon^2. \quad (3.52)$$

Adding the estimates (3.50) and (3.52), we have

$$J_{21} \leq C_1(1 + \epsilon^2\|(N_e, U)\|_\epsilon^4)\|(N_e, U)\|_\epsilon^2.$$

J_{16} and J_{20} are similar to J_{21} and can be bounded by $C_1(1 + \epsilon^2\|(N_e, U)\|_\epsilon^4)\|(N_e, U)\|_\epsilon^2$

Estimate of J_{22} . By using (2.17) and (2.18) in Lemma 2.4, similarly we have

$$J_{22} \leq C_1(1 + \epsilon^2\|(N_e, U)\|_\epsilon^4)\|(N_e, U)\|_\epsilon^2.$$

The proof of Lemma (3.4) is then complete. \square

3.5. Estimate of K_{12} . Next, we estimate K_{12} in (3.36).

Lemma 3.5 (Estimate of K_{12}). *Let (N_i, N_e, U) be a solution to (2.15), then there holds*

$$\begin{aligned} K_{12} \leq & -\frac{\epsilon}{2} \frac{d}{dt} \left[\int \frac{n_e}{n_i} (\partial_x^3 N_e)^2 + \epsilon \int \frac{n_e^2}{n_i} (\partial_x^4 N_e)^2 + \frac{\epsilon^2 H^2}{4} \int \frac{1}{n_i} (\partial_x^5 N_e)^2 \right] \\ & - \frac{\epsilon}{2} \frac{\epsilon H^2}{4} \frac{d}{dt} \left[\int \frac{1}{n_e n_i} (\partial_x^4 N_e)^2 + \epsilon \int \frac{1}{n_i} (\partial_x^5 N_e)^2 + \frac{\epsilon^2 H^2}{4} \int \frac{1}{(n_e)^2 n_i} (\partial_x^6 N_e)^2 \right] \\ & + C_1 (1 + \epsilon^2 \| (N_e, U) \|_\epsilon^6) (1 + \| (N_e, U) \|_\epsilon^2), \end{aligned}$$

Proof. Recall that in (3.36)

$$K_{12} = -\epsilon \int \left(\frac{n_e}{n_i} \partial_x^3 N_e - \frac{\epsilon H^2}{4} \frac{1}{n_e n_i} \partial_x^5 N_e \right) \partial_t \partial_x^3 N_i.$$

By the (2.15c), we have

$$\begin{aligned} \partial_t \partial_x^3 N_i &= \partial_t \partial_x^3 N_e - \epsilon \partial_t \partial_x^3 (n_e \partial_x^2 N_i) + \frac{\epsilon^2 H^2}{4} \partial_t \partial_x^3 \left(\frac{\partial_x^4 N_e}{n_e} \right) \\ &\quad - 2\epsilon^2 \partial_t \partial_x^3 (\partial_x \tilde{n}_e \partial_x N_e) - \epsilon^4 \partial_t \partial_x^3 ((\partial_x N_e)^2) - \epsilon^2 \partial_t \partial_x^3 (\partial_x^2 \tilde{n}_e N_e) \\ &\quad - \epsilon^2 \partial_t \partial_x^3 R_{(3)}^1 + \frac{H^2}{4} \left\{ -12\epsilon^5 \partial_t \partial_x^3 \left(\frac{(\partial_x \tilde{n}_e)^3}{n_e^4} \partial_x N_e \right) + 14\epsilon^4 \partial_t \partial_x^3 \left(\frac{\partial_x \tilde{n}_e \partial_x^2 \tilde{n}_e}{n_e^3} \partial_x N_e \right) \right. \\ &\quad - 3\epsilon^3 \partial_t \partial_x^3 \left(\frac{\partial_x^3 \tilde{n}_e}{n_e^2} \partial_x N_e \right) - 18\epsilon^7 \partial_t \partial_x^3 \left(\frac{(\partial_x \tilde{n}_e)^2}{n_e^4} (\partial_x N_e)^2 \right) + 7\epsilon^4 \partial_t \partial_x^3 \left(\frac{(\partial_x \tilde{n}_e)^2}{n_e^3} \partial_x^2 N_e \right) \\ &\quad + 7\epsilon^6 \partial_t \partial_x^3 \left(\frac{\partial_x^2 \tilde{n}_e}{n_e^3} (\partial_x N_e)^2 \right) - 4\epsilon^3 \partial_t \partial_x^3 \left(\frac{\partial_x^2 \tilde{n}_e}{n_e^2} \partial_x^2 N_e \right) - 12\epsilon^9 \partial_t \partial_x^3 \left(\frac{\partial_x \tilde{n}_e}{n_e^4} (\partial_x N_e)^3 \right) \\ &\quad + 14\epsilon^6 \partial_t \partial_x^3 \left(\frac{\partial_x \tilde{n}_e}{n_e^2} \partial_x N_e \partial_x^2 N_e \right) - 3\epsilon^3 \partial_t \partial_x^3 \left(\frac{\partial_x \tilde{n}_e}{n_e^2} \partial_x^3 N_e \right) - 3\epsilon^{11} \partial_t \partial_x^3 \left(\frac{1}{n_e^4} (\partial_x N_e)^4 \right) \\ &\quad + 7\epsilon^8 \partial_t \partial_x^3 \left(\frac{1}{n_e^3} (\partial_x N_e)^2 \partial_x^2 N_e \right) - 2\epsilon^5 \partial_t \partial_x^3 \left(\frac{1}{n_e^2} (\partial_x^2 N_e)^2 \right) - 3\epsilon^5 \partial_t \partial_x^3 \left(\frac{1}{n_e^2} \partial_x N_e \partial_x^3 N_e \right) \\ &\quad \left. + \epsilon^2 \partial_t \partial_x^3 \left(\frac{R_{(3)}^2 + R_{(3)}^3}{n_e^4} \right) \right\} \\ &=: \sum_{i=1}^{22} D_i. \end{aligned}$$

Accordingly, K_{12} is decomposed into

$$K_{12} = -\sum_{i=1}^{22} \epsilon \int \left(\frac{n_e}{n_i} \partial_x^3 N_e - \frac{\epsilon H^2}{4} \frac{1}{n_e n_i} \partial_x^5 N_e \right) D_i =: \sum_{i=1}^{22} I_i.$$

Estimate of I_1 .

$$I_1 = -\epsilon \int \left(\frac{n_e}{n_i} \partial_x^3 N_e - \frac{\epsilon H^2}{4} \frac{1}{n_e n_i} \partial_x^5 N_e \right) \partial_t \partial_x^3 N_e =: I_{11} + I_{12}.$$

By integration by parts, we have

$$\begin{aligned} I_{11} &= -\epsilon \int \frac{n_e}{n_i} \partial_x^3 N_e \partial_t \partial_x^3 N_e \\ &= -\frac{\epsilon}{2} \frac{d}{dt} \int \frac{n_e}{n_i} (\partial_x^3 N_e)^2 + \frac{\epsilon}{2} \int (\partial_t \frac{n_e}{n_i}) (\partial_x^3 N_e)^2. \end{aligned}$$

By direct computation, we have

$$\|\partial_t \frac{n_e}{n_i}\|_{L^\infty} \leq C(\epsilon + \epsilon^3(\|\partial_t N_e\|_{L^\infty} + \|\partial_t N_i\|_{L^\infty})). \quad (3.53)$$

Thus by Sobolev embedding $H^1 \hookrightarrow L^\infty$ and Lemma 3.2-3.3, we have

$$\begin{aligned} \frac{\epsilon}{2} \int (\partial_t \frac{n_e}{n_i})(\partial_x^3 N_e)^2 &\leq C \|\partial_t \frac{n_e}{n_i}\|_{L^\infty} (\epsilon \|\partial_x^3 N_e\|^2) \\ &\leq C_1(1 + \epsilon^2(\|\epsilon \partial_{tx} N_e\|^2 + \|\epsilon \partial_{tx} N_i\|^2))(\epsilon \|\partial_x^3 N_e\|^2) \\ &\leq C_1(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) \|(N_e, U)\|_\epsilon^2. \end{aligned}$$

Applying integration by parts again twice, we have

$$\begin{aligned} I_{12} &= -\frac{H^2}{4} \frac{\epsilon^2}{2} \frac{d}{dt} \int \frac{1}{n_e n_i} (\partial_x^4 N_e)^2 + \frac{H^2}{4} \frac{\epsilon^2}{2} \int \partial_t \left(\frac{1}{n_e n_i} \right) (\partial_x^4 N_e)^2 \\ &\quad - \frac{\epsilon^2 H^2}{4} \int \partial_x \left(\frac{1}{n_e n_i} \right) \partial_x^4 N_e \partial_t \partial_x^3 N_e \\ &=: I_{121} + I_{122} + I_{123}. \end{aligned}$$

Note that the estimate of $\|\partial_t(\frac{1}{n_e n_i})\|_{L^\infty}$ is similar to that for (3.53), thus similarly I_{122} can be estimated by $C_1(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) \|(N_e, U)\|_\epsilon^2$. By (3.37), Sobolev embedding theorem and Cauchy inequality, we have

$$\begin{aligned} I_{123} &\leq C(1 + \epsilon^2(\|\partial_x N_e\|_{L^\infty}^2 + \|\partial_x N_i\|_{L^\infty}^2))(\epsilon^2 \|\partial_x^4 N_e\|^2 + \epsilon^2 \|\epsilon \partial_t \partial_x^3 N_e\|^2) \\ &\leq C(1 + \epsilon^2(\|N_e\|_{H^2}^2 + \|N_i\|_{H^2}^2))(\epsilon^2 \|\partial_x^4 N_e\|^2 + \epsilon^2 \|\epsilon \partial_t \partial_x^3 N_e\|^2) \\ &\leq C_1(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) \|(N_e, U)\|_\epsilon^2, \end{aligned}$$

where we have used Lemma 3.1-3.3. Therefore, we obtain

$$\begin{aligned} I_1 &\leq -\frac{\epsilon}{2} \frac{d}{dt} \int \frac{n_e}{n_i} (\partial_x^3 N_e)^2 - \frac{H^2}{4} \frac{\epsilon^2}{2} \frac{d}{dt} \int \frac{1}{n_i n_e} (\partial_x^4 N_e)^2 \\ &\quad + C_1(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) \|(N_e, U)\|_\epsilon^2. \end{aligned} \quad (3.54)$$

Estimate of I_2 . Recall that

$$I_2 = \epsilon^2 \int \left(\frac{n_e}{n_i} \partial_x^3 N_e - \frac{\epsilon H^2}{4} \frac{1}{n_e n_i} \partial_x^5 N_e \right) \partial_t \partial_x^3 (n_e \partial_x^2 N_e) =: I_{21} + I_{22}.$$

Estimate of I_{21} . By integration by parts, we have

$$\begin{aligned} I_{21} &= \epsilon^2 \int \frac{n_e}{n_i} \partial_x^3 N_e \partial_t \partial_x^3 (n_e \partial_x^2 N_e) \\ &= -\epsilon^2 \int \frac{n_e}{n_i} \partial_x^4 N_e \partial_t \partial_x^2 (n_e \partial_x^2 N_e) - \epsilon^2 \int \left(\partial_x \frac{n_e}{n_i} \right) \partial_x^3 N_e \partial_t \partial_x^2 (n_e \partial_x^2 N_e) \\ &=: I_{211} + I_{212}. \end{aligned}$$

Estimate of I_{211} . By direct computation, we have

$$\begin{aligned} I_{211} &= -\epsilon^2 \int \frac{n_e^2}{n_i} \partial_x^4 N_e \partial_t \partial_x^4 N_e - \epsilon^2 \int \frac{n_e}{n_i} (\partial_x^4 N_e)^2 \partial_t n_e \\ &\quad - \epsilon^2 \int \frac{n_e}{n_i} \partial_x^4 N_e \partial_t (2 \partial_x n_e \partial_x^3 N_e + \partial_x^2 n_e \partial_x^2 N_e) \\ &=: I_{2111} + I_{2112} + I_{2113}. \end{aligned}$$

Note that the estimate of $\|\partial_t(n_e^2/n_i)\|_{L^\infty}$ is similar to (3.53), thus by integration by parts,

$$\begin{aligned} I_{2111} &= -\frac{\epsilon^2}{2} \frac{d}{dt} \int \frac{n_e^2}{n_i} (\partial_x^4 N_e)^2 + \frac{\epsilon^2}{2} \int \partial_t \left(\frac{n_e^2}{n_i} \right) (\partial_x^4 N_e)^2 \\ &\leq -\frac{\epsilon^2}{2} \frac{d}{dt} \int \frac{n_e^2}{n_i} (\partial_x^4 N_e)^2 + C_1 (1 + \epsilon^2 \| (N_e, U) \|_\epsilon^2) \| (N_e, U) \|_\epsilon^2. \end{aligned}$$

By Hölder inequality, Cauchy inequality, Sobolev embedding theorem and Lemma 3.1-3.3, we have

$$I_{2112} + I_{2113} \leq C_1 (1 + \epsilon^2 \| (N_e, U) \|_\epsilon^2) \| (N_e, U) \|_\epsilon^2.$$

By (3.37) and direct computation, we have

$$I_{212} \leq C_1 (1 + \epsilon^2 \| (N_e, U) \|_\epsilon^2) (1 + \| (N_e, U) \|_\epsilon^2).$$

Estimate of I_{22} . By direct computation, we have

$$\begin{aligned} I_{22} &= -\frac{\epsilon^3 H^2}{4} \int \frac{1}{n_e n_i} \partial_x^5 N_e \partial_t \partial_x^3 (n_e \partial_x^2 N_e) \\ &= -\frac{\epsilon^3 H^2}{4} \int \frac{1}{n_e n_i} \partial_x^5 N_e \partial_t (n_e \partial_x^5 N_e + \sum_{\beta=1}^3 C_3^\beta \partial_x^\beta n_e \partial_x^{5-\beta} N_e) \\ &= -\frac{\epsilon^3 H^2}{4} \int \frac{1}{n_i} \partial_x^5 N_e \partial_t \partial_x^5 N_e - \frac{\epsilon^3 H^2}{4} \int \frac{1}{n_e n_i} \partial_t n_e (\partial_x^5 N_e)^2 \\ &\quad - \frac{\epsilon^3 H^2}{4} \int \frac{\partial_x^5 N_e}{n_e n_i} \partial_t \left(\sum_{\beta=1}^3 C_3^\beta \partial_x^\beta n_e \partial_x^{5-\beta} N_e \right) \\ &=: I_{221} + I_{222} + I_{223}. \end{aligned}$$

By integration by parts in t , we have

$$I_{221} = -\frac{H^2}{4} \frac{\epsilon^2}{2} \frac{d}{dt} \int \frac{1}{n_i} (\partial_x^5 N_e)^2 + \frac{H^2}{4} \frac{\epsilon^2}{2} \int (\partial_x^5 N_e)^2 \partial_t \frac{1}{n_i}.$$

Note that

$$\left\| \partial_t \frac{1}{n_i} \right\|_{L^\infty} \leq C (\epsilon + \epsilon^3 \|\partial_t N_i\|_{L^\infty}). \quad (3.55)$$

Therefore, by Sobolev embedding theorem and Lemma 3.2, we have

$$\begin{aligned} \frac{H^2}{4} \frac{\epsilon^2}{2} \int (\partial_x^5 N_e)^2 \partial_t \frac{1}{n_i} &\leq C_1 (1 + \epsilon^2 \|\epsilon \partial_t N_i\|_{H^1}^2) (\epsilon^3 \|\partial_x^5 N_e\|^2) \\ &\leq C_1 (1 + \epsilon^2 \| (N_e, U) \|_\epsilon^2) \| (N_e, U) \|_\epsilon^2. \end{aligned}$$

Similarly, by Sobolev embedding theorem and Lemma 3.2-3.3, we have

$$I_{222} \leq C_1 (1 + \epsilon^2 \|\epsilon \partial_t N_e\|_{H^1}^2) (\epsilon^3 \|\partial_x^5 N_e\|^2) \leq C_1 (1 + \epsilon^2 \| (N_e, U) \|_\epsilon^2) \| (N_e, U) \|_\epsilon^2.$$

By Cauchy inequality, Sobolev embedding theorem and Lemma 3.1-3.3, we have

$$I_{223} \leq C_1 (1 + \epsilon^2 \| (N_e, U) \|_\epsilon^2) \| (N_e, U) \|_\epsilon^2.$$

Therefore, we have

$$\begin{aligned} I_2 &\leq -\frac{\epsilon^2}{2} \frac{d}{dt} \int \frac{n_e^2}{n_i} (\partial_x^4 N_e)^2 - \frac{H^2}{4} \frac{\epsilon^2}{2} \frac{d}{dt} \int \frac{1}{n_i} (\partial_x^5 N_e)^2 \\ &\quad + C_1 (1 + \epsilon^2 \| (N_e, U) \|_\epsilon^2) (1 + \| (N_e, U) \|_\epsilon^2). \end{aligned} \quad (3.56)$$

Estimate of I_3 .

$$I_3 = -\frac{\epsilon^3 H^2}{4} \int \left(\frac{n_e}{n_i} \partial_x^3 N_e - \frac{\epsilon H^2}{4} \frac{1}{n_e n_i} \partial_x^5 N_e \right) \partial_t \partial_x^3 \left(\frac{\partial_x^4 N_e}{n_e} \right) =: I_{31} + I_{32}.$$

Estimate of I_{31} . By integration by parts twice,

$$\begin{aligned} I_{31} &= -\frac{\epsilon^3 H^2}{4} \int \frac{n_e}{n_i} \partial_x^5 N_e \partial_{tx} \left(\frac{\partial_x^4 N_e}{n_e} \right) - \frac{\epsilon^3 H^2}{2} \int \partial_x^4 N_e \partial_x \left(\frac{n_e}{n_i} \right) \partial_{tx} \left(\frac{\partial_x^4 N_e}{n_e} \right) \\ &\quad - \frac{\epsilon^3 H^2}{4} \int \partial_x^3 N_e \partial_x^2 \left(\frac{n_e}{n_i} \right) \partial_{tx} \left(\frac{\partial_x^4 N_e}{n_e} \right) \\ &=: I_{311} + I_{312} + I_{313}. \end{aligned}$$

By direct computation, we have

$$\begin{aligned} I_{311} &= -\frac{\epsilon^3 H^2}{4} \int \frac{1}{n_i} \partial_x^5 N_e \partial_t \partial_x^5 N_e \\ &\quad - \frac{\epsilon^3 H^2}{4} \int \frac{n_e}{n_i} \partial_x^5 N_e \left(\left(\partial_t \frac{1}{n_e} \right) \partial_x^5 N_e + \left(\partial_t \frac{1}{n_e} \right) \partial_t \partial_x^4 N_e + \left(\partial_{tx} \frac{1}{n_e} \right) \partial_x^4 N_e \right) \\ &=: I_{3111} + I_{3112}. \end{aligned}$$

By integration by parts in t , Sobolev embedding, (3.55) and Lemma 3.2, we have

$$\begin{aligned} I_{3111} &= -\frac{1}{2} \frac{\epsilon^3 H^2}{4} \frac{d}{dt} \int \frac{1}{n_i} (\partial_x^5 N_e)^2 + \frac{\epsilon^3 H^2}{4} \int \partial_t \left(\frac{1}{n_i} \right) (\partial_x^5 N_e)^2 \\ &\leq -\frac{1}{2} \frac{\epsilon^3 H^2}{4} \frac{d}{dt} \int \frac{1}{n_i} (\partial_x^5 N_e)^2 + C_1 (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) \|(N_e, U)\|_\epsilon^2. \end{aligned}$$

Note that

$$\left\| \partial_t \frac{1}{n_e} \right\|_{L^\infty} \leq C(\epsilon + \epsilon^3 \|\partial_t N_e\|_{L^\infty}), \quad (3.57)$$

and

$$\left\| \partial_{tx} \frac{1}{n_e} \right\|_{L^\infty} \leq C(\epsilon + \epsilon^3 (\|\partial_t N_e\|_{L^\infty} + \|\partial_x N_e\|_{L^\infty}) + \epsilon^6 \|\partial_t N_e\|_{L^\infty} \|\partial_x N_e\|_{L^\infty}). \quad (3.58)$$

By Sobolev embedding $H^1 \hookrightarrow L^\infty$, Cauchy inequality and Lemma 3.2-3.3, we have

$$I_{3112} \leq C_1 (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) (1 + \|(N_e, U)\|_\epsilon^2).$$

Therefore, we obtain

$$I_{311} \leq -\frac{1}{2} \frac{\epsilon^3 H^2}{4} \frac{d}{dt} \int \frac{1}{n_i} (\partial_x^5 N_e)^2 + C_1 (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) (1 + \|(N_e, U)\|_\epsilon^2).$$

By direct computation, we have

$$\begin{aligned} I_{312} &= -\frac{\epsilon^3 H^2}{2} \int \partial_x^4 N_e \partial_x \left(\frac{n_e}{n_i} \right) \partial_{tx} \left(\frac{\partial_x^4 N_e}{n_e} \right) \\ &= -\frac{\epsilon^3 H^2}{2} \int \partial_x^4 N_e \partial_x \left(\frac{n_e}{n_i} \right) \left(\frac{1}{n_e} \partial_t \partial_x^5 N_e + \partial_t \frac{1}{n_e} \partial_x^5 N_e + \partial_x \frac{1}{n_e} \partial_t \partial_x^4 N_e + \partial_{tx} \frac{1}{n_e} \partial_x^4 N_e \right). \end{aligned}$$

By Sobolev embedding, Cauchy inequality and Lemma 3.2-3.3, we have

$$I_{312} \leq C_1 (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) (1 + \|(N_e, U)\|_\epsilon^2),$$

where we have used (3.37), (3.57) and (3.58). I_{313} is similar to I_{312} , thus we have

$$I_{31} \leq -\frac{1}{2} \frac{\epsilon^3 H^2}{4} \frac{d}{dt} \int \frac{1}{n_i} (\partial_x^5 N_e)^2 + C_1 (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) (1 + \|(N_e, U)\|_\epsilon^2). \quad (3.59)$$

Estimate of I_{32} . By integration by parts, we have

$$\begin{aligned} I_{32} &= \frac{\epsilon^3 H^2}{4} \frac{\epsilon H^2}{4} \int \frac{1}{n_e n_i} \partial_x^5 N_e \partial_t \partial_x^3 \left(\frac{\partial_x^4 N_e}{n_e} \right) \\ &= -\frac{\epsilon^3 H^2}{4} \frac{\epsilon H^2}{4} \int \frac{1}{n_e n_i} \partial_x^6 N_e \partial_t \partial_x^2 \left(\frac{\partial_x^4 N_e}{n_e} \right) - \frac{\epsilon^3 H^2}{4} \frac{\epsilon H^2}{4} \int \left(\partial_x \frac{1}{n_e n_i} \right) \partial_x^5 N_e \partial_t \partial_x^2 \left(\frac{\partial_x^4 N_e}{n_e} \right) \\ &=: I_{321} + I_{322}. \end{aligned}$$

By direct computation, we have

$$\begin{aligned} I_{321} &= -\frac{\epsilon^3 H^2}{4} \frac{\epsilon H^2}{4} \int \frac{1}{n_e n_i} \partial_x^6 N_e \partial_t \partial_x^6 N_e \\ &\quad - \frac{\epsilon^3 H^2}{4} \frac{\epsilon H^2}{4} \int \frac{1}{n_e n_i} \partial_x^6 N_e \left(\left(\partial_t \frac{1}{n_e} \right) \partial_x^6 N_e + 2 \left(\partial_{tx} \frac{1}{n_e} \right) \partial_x^5 N_e \right. \\ &\quad \left. + 2 \left(\partial_x \frac{1}{n_e} \right) \partial_t \partial_x^5 N_e + \left(\partial_x^2 \frac{1}{n_e} \right) \partial_t \partial_x^4 N_e + \left(\partial_t \partial_x^2 \frac{1}{n_e} \right) \partial_x^4 N_e \right) \\ &=: I_{3211} + I_{3212}. \end{aligned}$$

By integration by parts in t , we have

$$I_{3211} = -\frac{1}{2} \frac{\epsilon^3 H^2}{4} \frac{\epsilon H^2}{4} \frac{d}{dt} \int \frac{1}{n_e^2 n_i} (\partial_x^6 N_e)^2 + \frac{1}{2} \frac{\epsilon^3 H^2}{4} \frac{\epsilon H^2}{4} \int \partial_t \left(\frac{1}{n_e^2 n_i} \right) (\partial_x^6 N_e)^2.$$

Similar to (3.53), $\|\partial_t(1/n_e^2 n_i)\|_{L^\infty}$ has same bound with $\|\partial_t(n_e/n_i)\|_{L^\infty}$. Therefore, by Sobolev embedding $H^1 \hookrightarrow L^\infty$ and Lemma 3.2-3.3, we have

$$\begin{aligned} \frac{1}{2} \frac{\epsilon^3 H^2}{4} \frac{\epsilon H^2}{4} \int \partial_t \left(\frac{1}{n_e^2 n_i} \right) (\partial_x^6 N_e)^2 &\leq C(1 + \epsilon^2 (\|\epsilon \partial_t N_e\|_{L^\infty}^2 + \|\epsilon \partial_t N_i\|_{L^\infty}^2)) \epsilon^4 \|\partial_x^6 N_e\|^2 \\ &\leq C(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) \|(N_e, U)\|_\epsilon^2. \end{aligned}$$

By direct computation, we have

$$\begin{aligned} \|\partial_t \partial_x^2 \frac{1}{n_e}\|_{L^\infty} &\leq C(\epsilon + \epsilon^3 (\|\partial_x N_e\|_{L^\infty} + \|\partial_t N_e\|_{L^\infty} + \|\partial_t \partial_x^2 N_e\|_{L^\infty})) \\ &\quad + \epsilon^6 (\|\partial_x N_e\|_{L^\infty} \|\partial_t N_e\|_{L^\infty} + \|\partial_x N_e\|_{L^\infty}^2 + \|\partial_t N_e\|_{L^\infty} \|\partial_x^2 N_e\|_{L^\infty}) \\ &\quad + \epsilon^9 \|\partial_t N_e\|_{L^\infty} \|\partial_x N_e\|_{L^\infty}^2. \end{aligned} \quad (3.60)$$

By (3.26), (3.57), (3.58) and (3.60), we have

$$I_{3212} \leq C_1(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^4)(1 + \|(N_e, U)\|_\epsilon^2),$$

where we have used Sobolev embedding theorem, Lemma 3.2 and 3.3. Thus, we have

$$I_{321} \leq -\frac{1}{2} \frac{\epsilon^3 H^2}{4} \frac{\epsilon H^2}{4} \frac{d}{dt} \int \frac{1}{n_e^2 n_i} (\partial_x^6 N_e)^2 + C_1(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^4)(1 + \|(N_e, U)\|_\epsilon^2). \quad (3.61)$$

By integration by parts, we have

$$\begin{aligned} I_{322} &= -\frac{\epsilon^3 H^2}{4} \frac{\epsilon H^2}{4} \int \partial_x^5 N_e \left(\partial_x \frac{1}{n_e n_i} \right) \partial_t \partial_x^2 \left(\frac{\partial_x^4 N_e}{n_e} \right) \\ &= \frac{\epsilon^3 H^2}{4} \frac{\epsilon H^2}{4} \int \partial_x^6 N_e \left(\partial_x \frac{1}{n_e n_i} \right) \partial_t \partial_x \left(\frac{\partial_x^4 N_e}{n_e} \right) \\ &\quad + \frac{\epsilon^3 H^2}{4} \frac{\epsilon H^2}{4} \int \partial_x^5 N_e \left(\partial_x^2 \frac{1}{n_e n_i} \right) \partial_t \partial_x \left(\frac{\partial_x^4 N_e}{n_e} \right) \\ &=: I_{3221} + I_{3222}. \end{aligned}$$

By direct computation, we have

$$I_{3221} = \frac{\epsilon^3 H^2}{4} \frac{\epsilon H^2}{4} \int \partial_x^6 N_e (\partial_x \frac{1}{n_e n_i}) \left(\frac{1}{n_e} \partial_t \partial_x^5 N_e + (\partial_t \frac{1}{n_e}) \partial_x^5 N_e \right. \\ \left. + (\partial_x \frac{1}{n_e}) \partial_t \partial_x^4 N_e + (\partial_{tx} \frac{1}{n_e}) \partial_x^4 N_e \right).$$

By (3.4), (3.37), (3.57) and (3.58), Sobolev embedding theorem and Lemma 3.2-3.3, we have

$$I_{3221} \leq C_1 (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^4) (1 + \|(N_e, U)\|_\epsilon^2).$$

By direct computation, we have

$$\|\partial_x^2 (\frac{1}{n_e n_i})\|_{L^\infty} \leq C (\epsilon + \epsilon^3 (\|\partial_x N_e\|_{L^\infty} + \|\partial_x N_i\|_{L^\infty}) + \epsilon^6 (\|\partial_x N_e\|_{L^\infty} \|\partial_x N_i\|_{L^\infty} \\ + \|\partial_x N_e\|_{L^\infty}^2 + \|\partial_x N_i\|_{L^\infty}^2)). \quad (3.62)$$

Thus by (3.4), (3.57), (3.58) and (3.62), Sobolev embedding theorem and Lemma 3.1-3.3, we have

$$I_{3222} \leq C_1 (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^4) (1 + \|(N_e, U)\|_\epsilon^2).$$

Therefore, we have

$$I_{322} \leq C_1 (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^4) (1 + \|(N_e, U)\|_\epsilon^2). \quad (3.63)$$

Adding (3.61) and (3.63), we have

$$I_{32} \leq -\frac{1}{2} \frac{\epsilon^3 H^2}{4} \frac{\epsilon H^2}{4} \frac{d}{dt} \int \frac{1}{n_e^2 n_i} (\partial_x^6 N_e)^2 + C_1 (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^4) (1 + \|(N_e, U)\|_\epsilon^2). \quad (3.64)$$

Combining to (3.59) and (3.64), we have

$$I_3 \leq -\frac{1}{2} \frac{\epsilon^3 H^2}{4} \frac{d}{dt} \int \frac{1}{n_i} (\partial_x^5 N_e)^2 - \frac{1}{2} \frac{\epsilon^3 H^2}{4} \frac{\epsilon H^2}{4} \frac{d}{dt} \int \frac{1}{n_e^2 n_i} (\partial_x^6 N_e)^2 \\ + C_1 (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^4) (1 + \|(N_e, U)\|_\epsilon^2). \quad (3.65)$$

Thus combining (3.54), (3.56) and (3.65), we obtain

$$\sum_{i=1}^3 I_i \leq -\frac{1}{2} \frac{d}{dt} \left\{ \int \frac{n_e}{n_i} (\partial_x^2 N_e)^2 + \epsilon \int \frac{n_e^2}{n_i} (\partial_x^3 N_e)^2 + \frac{\epsilon^2 H^2}{4} \int \frac{1}{n_i} (\partial_x^4 N_e)^2 \right\} \\ - \frac{1}{2} \frac{\epsilon H^2}{4} \frac{d}{dt} \left\{ \int \frac{1}{n_e n_i} (\partial_x^3 N_e)^2 + \epsilon \int \frac{1}{n_i} (\partial_x^4 N_e)^2 + \frac{\epsilon^2 H^2}{4} \int \frac{1}{n_e^2 n_i} (\partial_x^5 N_e)^2 \right\} \\ + C_1 (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^4) (1 + \|(N_e, U)\|_\epsilon^2).$$

By Lemma 3.2-3.3, we have

$$I_{4,5,6,7} \leq C_1 (1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2) (1 + \|(N_e, U)\|_\epsilon^2).$$

Estimate of I_{18} . By direct computation, we have

$$I_{18} = \frac{3H^2}{4} \epsilon^{12} \int \left(\frac{n_e}{n_i} \partial_x^3 N_e - \frac{\epsilon H^2}{4} \frac{1}{n_e n_i} \partial_x^5 N_e \right) \partial_x^3 \left\{ \left(\partial_t \frac{1}{n_e^4} \right) (\partial_x N_e)^4 + 4 \frac{1}{n_e^4} (\partial_x N_e)^3 \partial_{tx} N_e \right\} \\ =: I_{181} + I_{182}.$$

Estimate of I_{181} . Using commutator notation (2.23), we have

$$\partial_x^3 \left(\partial_t \left(\frac{1}{n_e^4} \right) (\partial_x N_e)^4 \right) = [\partial_x^3, \partial_t \left(\frac{1}{n_e^4} \right)] (\partial_x N_e)^4 + \partial_t \left(\frac{1}{n_e^4} \right) \partial_x^3 ((\partial_x^4 N_e)^4).$$

By commutator estimate (2.24), we have

$$\|[\partial_x^3, \partial_t(\frac{1}{n_e^4})](\partial_x N_e)^4\| \leq \|\partial_{tx}(\frac{1}{n_e^4})\|_{L^\infty} \|\partial_x^2(\partial_x N_e)^4\| + \|\partial_t \partial_x^3(\frac{1}{n_e^4})\| \|\partial_x N_e\|_{L^\infty}^4.$$

Note that the estimate of $\|\partial_{tx}(\frac{1}{n_e^4})\|_{L^\infty}$ is similar to that for (3.58). We note that

$$\|\partial_x^2(\partial_x N_e)^4\| \leq C(\|\partial_x N_e\|_{L^\infty}^2 \|\partial_x^2 N_e\|_{L^\infty} \|\partial_x^2 N_e\| + \|\partial_x N_e\|_{L^\infty}^3 \|\partial_x^3 N_e\|), \quad (3.66)$$

and

$$\begin{aligned} \|\partial_t \partial_x^3(\frac{1}{n_e^4})\| &\leq C(\epsilon + \epsilon^3(\|\partial_x N_e\| + \|\partial_t N_e\| + \|\partial_x^2 N_e\| + \|\partial_{tx} N_e\| \\ &\quad + \|\partial_t \partial_x^2 N_e\| + \|\partial_t \partial_x^2 N_e\|) + \epsilon^6(\|\partial_x N_e\|^2 + \|\partial_t N_e\| \|\partial_x N_e\|_{L^\infty} \\ &\quad + \|\partial_x N_e\|_{L^\infty} \|\partial_{tx} N_e\| + \|\partial_t N_e\|_{L^\infty} \|\partial_x^2 N_e\| + \|\partial_t N_e\| \partial_x^3 N_e\| + \|\partial_{tx} N_e\|_{L^\infty} \|\partial_x^2 N_e\| \\ &\quad + \|\partial_x N_e\|_{L^\infty} \|\partial_t \partial_x^2 N_e\|) + \epsilon^9(\|\partial_x N_e\|_{L^\infty}^2 \|\partial_x N_e\| + \|\partial_t N_e\| \|\partial_x N_e\|_{L^\infty} \\ &\quad + \|\partial_t N_e\|_{L^\infty} \|\partial_x N_e\|_{L^\infty} \|\partial_x^2 N_e\| + \|\partial_{tx} N_e\| \|\partial_x N_e\|_{L^\infty}^2 \\ &\quad + \epsilon^{12} \|\partial_t N_e\|_{L^\infty} \|\partial_x N_e\|_{L^\infty}^2 \|\partial_x N_e\|). \end{aligned} \quad (3.67)$$

Thus by (3.58), (3.57), (3.66), (3.67) and (3.41), Sobolev embedding theorem and Lemma 3.2-3.3, we have

$$I_{181} \leq C_1(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^4)(1 + \|(N_e, U)\|_\epsilon^2).$$

Estimate of I_{182} . Using commutator notation (2.23), we have

$$\partial_x^3(\frac{1}{n_e^4}(\partial_x N_e)^3 \partial_{tx} N_e) = [\partial_x^3, \frac{1}{n_e^4}](\partial_x N_e)^3 \partial_{tx} N_e + \frac{1}{n_e^4} \partial_x^3((\partial_x N_e)^3 \partial_{tx} N_e).$$

By commutator estimate (2.24) of Lemma 2.6, we have

$$\begin{aligned} \|[\partial_x^3, \frac{1}{n_e^4}](\partial_x N_e)^3 \partial_{tx} N_e\| &\leq \|\partial_x(\frac{1}{n_e^4})\|_{L^\infty} \|\partial_x^2((\partial_x N_e)^3 \partial_{tx} N_e)\| \\ &\quad + \|\partial_x^3(\frac{1}{n_e^4})\| \|\partial_x N_e\|_{L^\infty}^3 \|\partial_{tx} N_e\|_{L^\infty}. \end{aligned}$$

By direct computation, we have

$$\begin{aligned} \|\partial_x^2((\partial_x N_e)^3 \partial_{tx} N_e)\| &\leq C(\|\partial_x N_e\|_{L^\infty} \|\partial_x^2 N_e\|_{L^\infty}^2 \|\partial_{tx} N_e\| \\ &\quad + \|\partial_x N_e\|_{L^\infty}^2 \|\partial_x^3 N_e\|_{L^\infty} \|\partial_{tx} N_e\| + \|\partial_x N_e\|_{L^\infty}^2 \|\partial_x^2 N_e\|_{L^\infty} \|\partial_t \partial_x^2 N_e\| \\ &\quad + \|\partial_x N_e\|_{L^\infty}^3 \|\partial_t \partial_x^3 N_e\|), \end{aligned} \quad (3.68)$$

and

$$\begin{aligned} \|\partial_x^3((\partial_x N_e)^3 \partial_{tx} N_e)\| &\leq C(\|\partial_x N_e\|_{L^\infty} \|\partial_x^2 N_e\|_{L^\infty} \|\partial_x^3 N_e\|_{L^\infty} \|\partial_{tx} N_e\| \\ &\quad + \|\partial_x^2 N_e\|_{L^\infty}^3 \|\partial_{tx} N_e\| + \|\partial_x N_e\|_{L^\infty} \|\partial_x^2 N_e\|_{L^\infty}^2 \|\partial_t \partial_x^2 N_e\| \\ &\quad + \|\partial_x N_e\|_{L^\infty}^2 \|\partial_x^4 N_e\|_{L^\infty} \|\partial_{tx} N_e\| + \|\partial_x N_e\|_{L^\infty}^2 \|\partial_x^3 N_e\|_{L^\infty} \|\partial_t \partial_x^2 N_e\| \\ &\quad + \|\partial_x N_e\|_{L^\infty}^2 \|\partial_x^2 N_e\|_{L^\infty} \|\partial_t \partial_x^3 N_e\| + \|\partial_x N_e\|_{L^\infty}^3 \|\partial_t \partial_x^4 N_e\|). \end{aligned} \quad (3.69)$$

Thus by (3.26), (3.28), (3.68) and (3.69), Sobolev embedding theorem and Lemma 3.2-3.3, we have

$$I_{182} \leq C_1(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^6)(1 + \|(N_e, U)\|_\epsilon^2).$$

Therefore, we have

$$I_{18} \leq C_1(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^6)(1 + \|(N_e, U)\|_\epsilon^2).$$

The estimates of $I_8 \sim I_{11}$, I_{13} and I_{15} are similar to that for I_{18} .

Estimate of I_{21} . By direct computation, we have

$$\begin{aligned} I_{21} &= \frac{3H^2}{4}\epsilon^6 \int \left(\frac{n_e}{n_i} \partial_x^3 N_e - \frac{\epsilon H^2}{4} \frac{1}{n_e n_i} \partial_x^5 N_e \right) \partial_x^3 \left\{ \left(\partial_t \frac{1}{n_e^2} \right) \partial_x N_e \partial_x^3 N_e + \frac{1}{n_e^2} \partial_t (\partial_x N_e \partial_x^3 N_e) \right\} \\ &=: I_{211} + I_{212}. \end{aligned}$$

Estimate of I_{211} . Using commutator notation (2.23), we have

$$\partial_x^3 \left(\left(\partial_t \frac{1}{n_e^2} \right) \partial_x N_e \partial_x^3 N_e \right) = [\partial_x^3, \partial_t \frac{1}{n_e^2}] \partial_x N_e \partial_x^3 N_e + (\partial_t \frac{1}{n_e^2}) \partial_x^3 (\partial_x N_e \partial_x^3 N_e).$$

By commutator estimate (2.24) in Lemma 2.6, we have

$$\begin{aligned} \| [\partial_x^3, \partial_t \frac{1}{n_e^2}] \partial_x N_e \partial_x^3 N_e \| &\leq \| \partial_{tx} (\frac{1}{n_e^2}) \|_{L^\infty} \| \partial_x^2 (\partial_x N_e \partial_x^3 N_e) \| \\ &\quad + \| \partial_t \partial_x^3 (\frac{1}{n_e^2}) \| \| \partial_x N_e \|_{L^\infty} \| \partial_x^3 N_e \|_{L^\infty}. \end{aligned}$$

Note that the estimate of $\| \partial_{tx} (\frac{1}{n_e^2}) \|_{L^\infty}$ is similar to that for (3.58). By direct computation, we note that

$$\| \partial_x^2 (\partial_x N_e \partial_x^3 N_e) \| \leq C (\| \partial_x^3 N_e \|_{L^\infty} \| \partial_x^3 N_e \| + \| \partial_x N_e \|_{L^\infty} \| \partial_x^5 N_e \| + \| \partial_x^2 N_e \|_{L^\infty} \| \partial_x^4 N_e \|), \quad (3.70)$$

and

$$\| \partial_x^3 (\partial_x N_e \partial_x^3 N_e) \| \leq C (\| \partial_x^3 N_e \|_{L^\infty} \| \partial_x^4 N_e \| + \| \partial_{x^2} N_e \|_{L^\infty} \| \partial_x^5 N_e \| + \| \partial_x N_e \|_{L^\infty} \| \partial_x^6 N_e \|). \quad (3.71)$$

Thus by (3.58), (3.67), (3.70), (3.71) and (3.57), Sobolev embedding theorem and Lemma 3.2-3.3, we have

$$I_{211} \leq C_1 (1 + \epsilon^2 \| (N_e, U) \|_\epsilon^6) (1 + \| (N_e, U) \|_\epsilon^2).$$

Estimate of I_{212} . Using commutator notation (2.23), we have

$$\begin{aligned} I_{212} &= \frac{3H^2}{4}\epsilon^6 \int \left(\frac{n_e}{n_i} \partial_x^3 N_e - \frac{\epsilon H^2}{4} \frac{1}{n_e n_i} \partial_x^5 N_e \right) \partial_x^3 \left\{ \frac{1}{n_e^2} \partial_t (\partial_x N_e \partial_x^3 N_e) \right\} \\ &= \frac{3H^2}{4}\epsilon^6 \int \left(\frac{n_e}{n_i} \partial_x^3 N_e - \frac{\epsilon H^2}{4} \frac{1}{n_e n_i} \partial_x^5 N_e \right) \left\{ [\partial_x^3, \frac{1}{n_e^2}] \partial_t (\partial_x N_e \partial_x^3 N_e) \right. \\ &\quad \left. + \frac{1}{n_e^2} \partial_t \partial_x^3 (\partial_x N_e \partial_x^3 N_e) \right\} \\ &=: I_{2121} + I_{2122}. \end{aligned}$$

By commutator estimate (2.24) in Lemma 2.6, we have

$$\begin{aligned} \| [\partial_x^3, \frac{1}{n_e^2}] \partial_t (\partial_x N_e \partial_x^3 N_e) \| &\leq \| \partial_x (\frac{1}{n_e^2}) \|_{L^\infty} \| \partial_x^2 (\partial_t (\partial_x N_e \partial_x^3 N_e)) \| \\ &\quad + \| \partial_x^3 (\frac{1}{n_e^2}) \| \| \partial_t (\partial_x N_e \partial_x^3 N_e) \|_{L^\infty}. \end{aligned}$$

By direct computation, we have

$$\| \partial_t (\partial_x N_e \partial_x^3 N_e) \|_{L^\infty} \leq C (\| \partial_{tx} N_e \|_{L^\infty} \| \partial_x^3 N_e \|_{L^\infty} + \| \partial_x N_e \|_{L^\infty} \| \partial_t \partial_x^3 N_e \|_{L^\infty}), \quad (3.72)$$

and

$$\begin{aligned} \|\partial_t \partial_x^2 (\partial_x N_e \partial_x^3 N_e)\| &\leq C(\|\partial_x N_e\|_{L^\infty} \|\partial_t \partial_x^5 N_e\| + \|\partial_x^2 N_e\|_{L^\infty} \|\partial_t \partial_x^4 N_e\| \\ &\quad + \|\partial_x^3 N_e\|_{L^\infty} \|\partial_t \partial_x^3 N_e\| + \|\partial_x^4 N_e\|_{L^\infty} \|\partial_t \partial_x^2 N_e\| \\ &\quad + \|\partial_x^5 N_e\|_{L^\infty} \|\partial_t \partial_x N_e\|). \end{aligned} \quad (3.73)$$

Thus by (3.26), (3.41), (3.72) and (3.73), Sobolev embedding theorem and Lemma 3.2-3.3,

$$I_{2121} \leq C_1(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^4)(1 + \|(N_e, U)\|_\epsilon^2).$$

By integration by parts, we have

$$\begin{aligned} I_{2122} &= -\frac{3H^2}{4}\epsilon^6 \int \left\{ \partial_x \left(\frac{1}{n_e n_i} \right) \partial_x^3 N_e - \frac{\epsilon H^2}{4} \partial_x \left(\frac{1}{n_e^3 n_i} \right) \partial_x^5 N_e \right\} \partial_t \partial_x^2 (\partial_x N_e \partial_x^3 N_e) \\ &\quad - \frac{3H^2}{4}\epsilon^6 \int \left\{ \frac{1}{n_e n_i} \partial_x^4 N_e - \frac{\epsilon H^2}{4} \frac{1}{n_e^3 n_i} \partial_x^6 N_e \right\} \partial_t \partial_x^2 (\partial_x N_e \partial_x^3 N_e). \end{aligned}$$

By direct computation, we note that $\partial_x \frac{1}{n_e^3 n_i}$ has same estimate with (3.37), thus by (3.73), Sobolev embedding theorem and Lemma 3.1-3.3, we have

$$I_{2122} \leq C_1(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^2)(1 + \|(N_e, U)\|_\epsilon^2).$$

Therefore, we have

$$I_{212} \leq C_1(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^4)(1 + \|(N_e, U)\|_\epsilon^2),$$

and hence

$$I_{21} \leq C_1(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^6)(1 + \|(N_e, U)\|_\epsilon^2).$$

The estimates of I_{12} , I_{14} , I_{16} , I_{17} , I_{19} and I_{20} are similar to that of I_{18} . According to the Lemma 2.4, we have

$$I_{22} \leq C_1(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^4)(1 + \|(N_e, U)\|_\epsilon^2).$$

The proof of Lemma 3.5 is then complete. \square

4. PROOF OF THEOREM 2.5

Proof of Theorem 2.5 . Adding Propositions 3.1 with $\gamma = 0, 1, 2$ and Proposition 3.2 together, we obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} (\|U\|_{H^2}^2 + \epsilon \|\partial_x^3 U\|_{L^2}^2) + \frac{1}{2} \frac{d}{dt} \left\{ \int \frac{n_e}{n_i} \left(\sum_{i=0}^2 |\partial_x^i N_e|^2 + \epsilon |\partial_x^3 N_e|^2 \right) \right\} \\ &\quad + \frac{1}{2} \frac{d}{dt} \left\{ \int \epsilon \left(\frac{n_e^2}{n_i} + \frac{H^2}{8n_e n_i} \right) \left(\sum_{i=0}^3 |\partial_x^i N_e|^2 + \epsilon |\partial_x^4 N_e|^2 \right) \right\} \\ &\quad + \frac{3H^2}{16} \frac{d}{dt} \left\{ \int \frac{\epsilon^2}{n_i} \left(\sum_{i=0}^4 |\partial_x^i N_e|^2 + \epsilon |\partial_x^5 N_e|^2 \right) \right\} \\ &\quad + \frac{H^2}{8} \frac{d}{dt} \left\{ \int \frac{\epsilon^3}{n_e^2 n_i} \left(\sum_{i=0}^5 |\partial_x^i N_e|^2 + \epsilon |\partial_x^6 N_e|^2 \right) \right\} \\ &\leq C(1 + \epsilon^2 \|(N_e, U)\|_\epsilon^6)(1 + \|(N_e, U)\|_\epsilon^2). \end{aligned} \quad (4.1)$$

Integrating the inequality (4.1) over $(0, t)$ yields

$$\begin{aligned} \|(N_e, U)(t)\|_\epsilon^2 &\leq C\|(N_e, U)(0)\|_\epsilon^2 + \int_0^t C_1(1 + \epsilon^2\|(N_e, U)\|_\epsilon^6)(1 + \|(N_e, U)\|_\epsilon^2)ds \\ &\leq C_1\|(N_e, U)(0)\|_\epsilon^2 + \int_0^t C_1(1 + \epsilon^2\tilde{C})(1 + \|(N_e, U)\|_\epsilon^2)ds, \end{aligned}$$

where C is an absolute constant.

Recall that C_1 depends on $\|(N_e, U)\|_\epsilon^2$ through $\epsilon\|(N_e, U)\|_\epsilon^2$ and is nondecreasing. Let $C'_1 = C_1(1)$ and $C_2 > C \sup_{\epsilon \leq 1} \|(u_R^\epsilon, \phi_R^\epsilon)(0)\|_\epsilon^2$. For any arbitrarily given $\tau > 0$, we choose \tilde{C} sufficiently large such that $\tilde{C} > e^{4C'_1\tau}(1 + C_2)(1 + C'_1)$. Then there exists $\epsilon_0 > 0$ such that $\epsilon\tilde{C} \leq 1$ for all $\epsilon < \epsilon_0$, we have

$$\sup_{0 \leq t \leq \tau} \|(N_e, U)(t)\|_\epsilon^2 \leq e^{4C'_1\tau}(C_2 + 1) < \tilde{C}. \quad (4.2)$$

In particular, we have the uniform bound for (N_e, U) ,

$$\begin{aligned} \sup_{0 \leq t \leq \tau} \left(\|(N_e, U)\|_{H^2}^2 + \epsilon\|(\partial_x^3 N_e, \partial_x^3 U)\|_{L^2}^2 \right. \\ \left. + \epsilon^2\|\partial_x^4 N_e\|_{L^2}^2 + \epsilon^3\|\partial_x^5 N_e\|_{L^2}^2 + \epsilon^4\|\partial_x^6 N_e\|_{L^2}^2 \right) \leq \tilde{C}. \end{aligned} \quad (4.3)$$

On the other hand, by Lemma 3.1 and (4.3), we have

$$\sup_{0 \leq t \leq \tau} \|N_i\|_{H^2}^2 \leq \tilde{C}.$$

It is now standard to obtain uniform estimates independent of ϵ by the continuity method. \square

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